FIELD THEORETIC APPROACH TO GRAVITY

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by
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Dedicated to my parents

PUBLICATIONS

- **Different types of torsion and their effect on the dynamics of fields**.
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- **Geometrical contribution to neutrino mass matrix**.
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- **Weak field limit in vierbein-Einstein-Palatini formalism and Fierz-Pauli Equation**.
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LIST OF SYMBOLS

1. $g_{\mu\nu}$: metric tensor
2. g: determinant of $g_{\mu\nu}$
3. η_{IJ}: Minkowski metric
4. $\widehat{\nabla}_{\mu}$: Levi-Civita connection
5. ∇_{μ}: covariant derivative with non-zero torsion
6. $\widehat{\Gamma}^{\alpha}{}_{\nu\mu}$: Christoffel symbols
7. $\Gamma^{\alpha}{}_{\mu\nu}$: general affine connection with torsion
8. $\widehat{R}^{\rho}{}_{\sigma\mu\nu}$: torsion-free Riemann tensor
9. $\widehat{R}_{\mu\nu}$: torsion-free Ricci tensor
10. $\widehat{R}$: torsion-free Ricci scalar
11. $\widehat{G}_{\mu\nu}(\equiv \widehat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\widehat{R})$: torsion-free Einstein tensor
12. $R^{\rho}{}_{\sigma\mu\nu}$: torsionful Riemann tensor
13. $R_{\mu\nu}$: torsionful Ricci tensor
14. R: torsionful Ricci scalar
15. $G_{\mu\nu}(\equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)$: torsionful Einstein tensor
16. $C^{\alpha}{}_{\mu\nu}$: (Cartan) torsion tensor
17. $S^{\alpha}{}_{\mu\nu}$: contorsion tensor
18. e^{I}_{μ}: tetrads
19. e^{μ}_{I}: co-tetrads
20. $|e|$: determinant of e^{I}_{μ}
21. A^{IJ}_{μ}: torsionful spin connection
22. Λ^{IJ}_{μ}: contorsion component of A^{IJ}_{μ}
23. D_{μ}: covariant derivative corresponding to A^{IJ}_{μ}
24. ω^{IJ}_{μ}: torsion-free spin connection
25. $\widehat{D}_{\mu}$: covariant derivative corresponding to ω^{IJ}_{μ}
26. $F^{IJ}_{\mu\nu}$: curvature of D_{μ}
27. $\widehat{F}^{IJ}_{\mu\nu}$: curvature of $\widehat{D}_{\mu}$

28. $\kappa = 8\pi G$
29. $S[e, A]$: action for gravity in vierbein-Einstein-Palatini formalism
30. $S_{tetrad}[e]$: action for gravity in tetrad formulation of GR
31. S_M: action for matter field
32. $\widehat{T}_{\mu\nu}$: symmetric and conserved energy-momentum tensor
33. $\Theta_{\mu\nu}$: non-symmetric energy-momentum tensor
34. ϕ: real scalar field
35. A_μ: Electromagnetic gauge field
36. $F_{\mu\nu}$: Electromagnetic field tensor
37. ψ: Dirac fermionic field
38. $\bar{\psi}$: Dirac adjoint of ψ
39. ${}^{\psi}\widehat{D}_\mu$: torsion-free covariant derivative of ψ
40. ${}^{\psi}D_\mu$: torsionful covariant derivative of ψ
41. γ_I: Dirac gamma matrices
42. $\mathcal{L}_F$: Lagrangian for Dirac spinor
43. $\mathcal{L}_\phi$: Lagrangian for scalar field
44. J^μ: probability current
45. Ω: conformal factor
46. ${}^{NY}C^{\alpha}{}_{\mu\nu}$: Nieh-Yan torsion
47. ${}^{Inv}C^{\alpha}{}_{\mu\nu}$: conformally invariant torsion
48. ${}^{OS}C^{\alpha}{}_{\mu\nu}$: on-shell torsion
49. ξ: conformal parameter for A_μ^{IJ}
50. $\Psi_{eL} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$: left-handed electron-neutrino doublet
51. e_R: right-handed electron singlet
52. $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$: Higgs doublet
53. ν_e: electron neutrino
54. ν_μ: muon neutrino
55. ν_l: neutrino of lepton l
56. $P_{\nu_{l'}\nu_l}(t)$: probability of finding a $\nu_{l'}$ at time t in a beam that started as ν_l
57. λ_f: coupling constant for fermion-torsion coupling

58. $\mathcal{M}$: forward scattering amplitude
59. $\mathscr{H}_{\text{eff}}$: effective Hamiltonian density
60. n_f: number density of fermions
61. ρ: weighted density
62. U: mixing matrix
63. P_{conv}: probability of conversion
64. $\mathscr{L}_{\text{cc}}$: effective Lagrangian due to charge current interaction
65. V_{nc}: contribution due to neutral current interaction in the Hamiltonian
66. G_F: Fermi constant
67. I^f_{3L}: third component of weak isospin for the left-handed component
68. Q_f: charge of fermion
69. θ: mixing angle
70. $\tilde{\theta}$: effective mixing angle
71. $h_{\mu\nu}$: metric perturbation
72. f^μ_I: co-tetrad perturbation
73. $\tilde{f}^I_\mu$: tetrad perturbation
74. a^{IJ}_μ: spin connection perturbation
75. $\mathcal{F}^{IJ}_{\mu\nu}$: perturbation of $F^{IJ}_{\mu\nu}$
76. $S = \bar{\psi}\psi$: scalar
77. $P = i\bar{\psi}\gamma_5\psi$: pseudoscalar
78. $V_I = i\bar{\psi}\gamma_I\psi$: vector
79. $A_I = i\psi\gamma_5\gamma_I\psi$: axial vector
80. $T_{IJ} = \bar{\psi}\sigma_{IJ}\psi$: tensor

CONVENTIONS

1. Metric signature $\quad (-+++)$
2. $\{A, B\} = AB + BA$
3. $[A, B] = AB - BA$
4. $A_{\{\mu}B_{\nu\}} = A_\mu B_\nu + A_\nu B_\mu$
5. $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$
6. $A_{\{\mu|\alpha|}B_{\nu\}} = A_{\mu\alpha}B_\nu + A_{\nu\alpha}B_\mu$
7. $A_{[\mu|\alpha|}B_{\nu]} = A_{\mu\alpha}B_\nu - A_{\nu\alpha}B_\mu$
8. $A_{\{\mu}B^{\alpha}{}_{\nu\}} = A_\mu B^\alpha{}_\nu + A_\nu B^\alpha{}_\mu$
9. $A_{[\mu}B^{\alpha}{}_{\nu]} = A_\mu B^\alpha{}_\nu - A_\nu B^\alpha{}_\mu$
10. $\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta + \Gamma^\alpha{}_{\mu\nu} T^\nu{}_\beta - \Gamma^\nu{}_{\mu\beta} T^\alpha{}_\nu$
11. $D_\mu T^I{}_J = \partial_\mu T^I{}_J + A^I_{\mu K} T^K{}_J - A^K_{\mu J} T^I{}_K$

CONTENTS

1

INTRODUCTION

Conventionally, General Relativity is formulated purely from a metric ($g_{\mu\nu}$) point of view. Unlike other field theories, General Relativity does not have an independent connection and also it is a second order theory. Although we work with the Levi-Civita connection ∇, the corresponding affine connection Γ is not an independent quantity because of the following two assumptions

1. Metric compatibility:

$$\nabla_\alpha g_{\mu\nu} = 0\,, \tag{1.1}$$

2. Torsion-free condition:

$$\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}\,. \tag{1.2}$$

The above allow us to write the affine connection as the Christoffel symbols $\widehat{\Gamma}$,

$$\Gamma^\alpha{}_{\mu\nu} \equiv \widehat{\Gamma}^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)\,. \tag{1.3}$$

The above assumptions imply that Levi-Civita connection is unique torsion-free connection in Riemannian geometry. I will discuss the importance of the above two assumptions as to why and when they are justified. The metric compatibility or simply metricity is a necessity rather than a mere assumption. We want to have quantities like inner product of vectors, length element invariant under parallel transport. The derivative operator ∇_μ defines the parallel transport of a vector v^μ along a curve with tangent t^μ by

$$t^\mu \nabla_\mu v^\nu = 0\,. \tag{1.4}$$

We want the inner product $g_{\mu\nu}v^\mu u^\nu$ of two vectors v^μ and u^μ to remain unaltered under parallel transport i.e.,

$$\begin{aligned} & t^\lambda \nabla_\lambda (g_{\mu\nu} v^\mu u^\nu) = 0 \\ \Rightarrow \quad & t^\lambda v^\mu u^\nu \nabla_\lambda g_{\mu\nu} = 0\,, \end{aligned} \tag{1.5}$$

where, I have used Eq. (1.4) in obtaining the last equation above. This means that the metric must remain invariant under parallel transport. It is worth mentioning that there have been some works in non-metricity [1] but I will not cover those in this thesis.

The second assumption of torsion-free connection is not necessary and it depends on the matter fields present. In general for bosonic matter torsion does not have any effect on the equations and it vanishes on-shell. As we will see, for fermionic matter, the above is not the case. Thus, although in General Relativity torsion is set to zero, it is always interesting to consider a more general theory with non-zero torsion. The first attempt to formulate a theory of gravity that included torsion was made by Cartan [2]. Later Einstein attempted to match torsion with electromagnetic field tensor in search of a unified theory although he was unsuccessful. This is known as Einstein-Cartan theory. It was further developed by Kibble [3], Sciama [4], and later Hehl [5], relating torsion to the spin angular momentum of matter, in particular fermionic matter.

The torsion-less limit of torsion gravity cannot always be taken continuously [6, 7]. In this thesis we will be looking primarily at a torsion theory of gravity which in the torsion-free limit reduces to Einstein's General Relativity. One way of introducing the torsion field into the theory as a dynamical variable would be to add it in as the antisymmetric part of the connection coefficients (Cartan torsion).

$$C^\rho{}_{\nu\mu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}\,. \tag{1.6}$$

I will find out the form of curvature tensor for connection with torsion. In General Relativity without torsion, the Riemann curvature $R(X,Y)Z$ is defined as

$$[\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = R(X,Y)Z\,. \tag{1.7}$$

Taking basis vector fields $\partial_\mu, \partial_\nu$ for X, Y respectively and v^ρ for Z gives

$$(\widehat{\nabla}_\mu \widehat{\nabla}_\nu - \widehat{\nabla}_\nu \widehat{\nabla}_\mu)v^\rho = \widehat{R}^\rho{}_{\sigma\mu\nu}[\widehat{\Gamma}]v^\sigma\,. \tag{1.8}$$

I will write quantities corresponding to torsion-free connection with a '$\widehat{\ }$' above. In terms of Christoffel symbols the Riemann tensor of Eq. (1.8) is given by

$$\widehat{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \widehat{\Gamma}^\rho{}_{\nu\sigma} - \partial_\nu \widehat{\Gamma}^\rho{}_{\mu\sigma} + \widehat{\Gamma}^\rho{}_{\mu\lambda}\widehat{\Gamma}^\lambda{}_{\nu\sigma} - \widehat{\Gamma}^\rho{}_{\nu\lambda}\widehat{\Gamma}^\lambda{}_{\mu\sigma}\,. \tag{1.9}$$

Now if ∇ is a general affine connection with torsionful Γ, the definition of the Riemann tensor is modified to

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)v^\rho = R^\rho{}_{\sigma\mu\nu}[\Gamma]v^\sigma - C^\sigma{}_{\nu\mu}\nabla_\sigma v^\rho\,. \tag{1.10}$$

The definition of the Riemann tensor thus has to be modified in case of non-zero torsion as

$$R^\rho{}_{\sigma\mu\nu}[\Gamma]v^\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)v^\rho + C^\sigma{}_{\nu\mu}\nabla_\sigma v^\rho\,. \tag{1.11}$$

Let us look at the symmetries and identities of the Riemann tensor with Cartan torsion and compare them with those in General Relativity.

1.0.1 *Skew symmetry*

The antisymmetry of the Riemann tensor in each pair is there in both General Relativity and with non-zero torsion.

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu}\,. \tag{1.12}$$

1.0.2 *Pairwise symmetry*

Unlike in General Relativity, where the Riemann tensor is symmetric under pair exchange, with non-zero torsion the symmetry gets lost. The antisymmetry under the exchange of pairs is given by (see Eq. (A.19))

$$\begin{aligned} R_{\rho\sigma\mu\nu} - R_{\mu\nu\rho\sigma} =& \frac{1}{2}\Big(\nabla_{[\rho}C_{|\nu|\sigma\mu]} + C_{\nu\lambda[\rho}C^{\lambda}{}_{\sigma\mu]} + \nabla_{[\sigma}C_{|\rho|\mu\nu]} + C_{\rho\lambda[\sigma}C^{\lambda}{}_{\mu\nu]} \\ & -\nabla_{[\mu}C_{|\sigma|\nu\rho]} - C_{\sigma\lambda[\mu}C^{\lambda}{}_{\nu\rho]} - \nabla_{[\nu}C_{|\mu|\rho\sigma]} - C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]}\Big) \,. \end{aligned} \tag{1.13}$$

In fact the Ricci tensor which is symmetric in General Relativity, is not so in presence of torsion.

1.0.3 *First Bianchi identity*

The first or algebraic Bianchi identity gets modified as (see Eq. (A.16))

$$R^{\rho}{}_{[\sigma\mu\nu]} = \nabla_{[\sigma}C^{\rho}{}_{\mu\nu]} + C^{\rho}{}_{\lambda[\sigma}C^{\lambda}{}_{\mu\nu]} \,, \tag{1.14}$$

1.0.4 *Second Bianchi identity*

The second or differential Bianchi identity also gets modified as (see Eq. (A.24))

$$\nabla_{[\lambda}R^{\rho}{}_{|\sigma|\mu\nu]} = -R^{\rho}{}_{\sigma\alpha[\lambda}C^{\alpha}{}_{\mu\nu]} \,. \tag{1.15}$$

Thus three out of the four identities of the Riemann tensor get changed in case of non-zero torsion. A detailed calculation of the above identities have been given in Appendix A.

It is interesting to note that with the help of metric compatibility, the affine connection can be decomposed in terms of the Christoffel symbols and the so called *contorsion tensor*.

$$\Gamma^{\alpha}{}_{\mu\nu} = \widehat{\Gamma}^{\alpha}{}_{\mu\nu} - S^{\alpha}{}_{\mu\nu} \,, \tag{1.16}$$

where the contorsion tensor $S^{\alpha}{}_{\mu\nu}$ is given by (see Eq. (A.8))

$$S^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(C_{\mu\nu}{}^{\alpha} + C_{\nu\mu}{}^{\alpha} - C^{\alpha}{}_{\nu\mu}\right) . \tag{1.17}$$

Although torsion can be considered as an extension to General Relativity by adding contorsion tensor to the Christoffel symbols, the way to couple torsion to other fields, particularly to fermions, is not obvious in this approach. A more transparent and geometrical way of introducing torsion is to work with the first order Palatini formulation of gravity, using local orthogonal coordinates or frame fields called tetrads or vierbeins, and a local Lorentz connection called the spin connection [3, 8, 9]. I will call this vierbein-Einstein-Palatini formalism. In the usual Palatini formalism, the metric and the connection are considered to be independent variables off-shell and when gravity is coupled to only bosonic fields, this formalism reduces to the usual metric formalism of General Relativity on-shell where the connection is given by the Christoffel symbols. In vierbein-Einstein-Palatini formalism the above corresponds to writing the spin connection in terms of the tetrads and their derivatives. We will see later that if there are fermionic fields contributing to the stress-energy tensor, the spin connection has torsion components and remains independent. The vierbein-Einstein-Palatini formalism is particularly useful for writing a Lagrangian for fermionic fields on curved spacetime [10–12], as it highlights the spin connection as being analogous to a gauge field. In addition, this formalism serves as the link between General Relativity and BF theories of gravity [13–16]. The Einstein-Hilbert action does not resemble that of other field theories. Often we are interested in fields that are functions of flat spacetime coordinates and this is where vierbein-Einstein-Palatini formalism comes handy. Because of the fact that we have an independent spin connection that resembles a gauge field, vierbein-Einstein-Palatini formalism looks like a field theory of gravity.

1.1 SYNOPSIS AND PLAN OF THESIS

In this thesis I will explore different aspects of the vierbein-Einstein-Palatini formalism. In chapter 2 I will go through the vierbein-Einstein-Palatini formalism. I will define the variables, write the action and find out the equations of motion. I will compare the results with those in General Relativity. In chapter 3 I will discuss the vierbein-Einstein-Palatini formalism in the presence of bosonic and fermionic matter. I will demonstrate how the presence of fermions yields non-zero torsion thereby leading to a departure from General Relativity. In chapter 4 I will discuss the conformal properties of the vierbein-Einstein-Palatini variables and see how torsion affects the conformal invariance of fields. In chapter 5 I will use an interesting result of the four-fermion interaction that results from spin-torsion coupling to answer the source of neutrino mass and oscillation. Finally in chapter 6 I will consider the perturbations of the vierbein-Einstein-Palatini variables around arbitrary background and find general perturbation equation.

2

VIERBEIN-EINSTEIN-PALATINI FORMALISM

In the vierbein-Einstein-Palatini formalism, the variables for gravity are the vierbein or tetrads e^I_μ, and the spin connection A^{IJ}_μ. One considers local 4d flat space (internal space) at each point of spacetime manifold isomorphic to the tangent space at that point. Linear isomorphisms between vector fields and sections of internal space are given by tetrads. Tetrads can be thought of as frame fields. I will denote spacetime indices by lowercase Greek letters and internal indices by uppercase Roman letters. The internal space is a 4-dimensional flat space with metric $\eta_{IJ} = (-1, 1, 1, 1)$ attached to each point of spacetime. Raising and lowering of the internal indices are done by η, while spacetime indices are raised and lowered by the spacetime metric g, which is also of signature $(- + ++)$. I can write the basis vector field of the spacetime manifold λ_μ as

$$\lambda_\mu = e^I_\mu \zeta_I \,, \tag{2.1}$$

where ζ_I are the internal basis vectors with I running from 0 to 3. Although written in terms of basis vectors fields, the above holds for all vector fields. The tetrads are considered to be orthonormal,

$$g^{\mu\nu} e^I_\mu e^J_\nu = \eta^{IJ} \,. \tag{2.2}$$

This equation can also be thought of as a relation between the spacetime metric and the internal metric. Clearly, tetrads contain the same information as the spacetime metric and constitute the main variable in the vierbein-Einstein-Palatini formalism. Eq. (2.2) can be rewritten as

$$e^I_\mu e^\mu_J = \delta^I_J \,, \qquad e^\mu_I e^I_\nu = \delta^\mu_\nu \,, \tag{2.3}$$

where $e^\mu_I \equiv \eta_{IJ} g^{\mu\nu} e^J_\nu$ are the inverse of tetrads, called co-tetrads. It is easy to see that the determinants of the tetrad and the metric are related by $|e| = \sqrt{-g}$.

A connection D on the frame bundle is defined by its action on any smooth section S,

$$D_\mu S^I = \partial_\mu S^I + A^I_{\mu J} S^J \,, \tag{2.4}$$

where $A^I_{\mu J}$ are the components of what is called the spin connection. It follows from definition that A^{IJ}_μ is antisymmetric in the internal indices,

$$\begin{aligned} 0 &= D_\mu \eta^{IJ} = \partial_\mu \eta^{IJ} - A^I_{\mu K} \eta^{KJ} - A^J_{\mu K} \eta^{IK} \\ \Rightarrow \quad A^{IJ}_\mu &= -A^{JI}_\mu . \end{aligned} \tag{2.5}$$

The curvature of D can be written as

$$\begin{aligned} F^{IJ}_{\mu\nu} &= \left[D_\mu \,, D_\nu \right]^{IJ} \\ &= \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + A^I_{\mu K} A^{KJ}_\nu - A^I_{\nu K} A^{KJ}_\nu \\ &= \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + [A_\mu, A_\nu]^{IJ} \,. \end{aligned} \tag{2.6}$$

I want to define a general affine connection with the help of the spin connection. The simplest way is to define the connection as

$$\Gamma^\alpha{}_{\mu\nu} = A^I_{\mu J} e^J_\nu e^\alpha_I \,. \tag{2.7}$$

The action of the corresponding covariant derivative is given by

$$\begin{aligned} \nabla_\mu v^\nu =& \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha \\ =& \partial_\mu v^\nu + A^I_{\mu J} e^J_\alpha e^\nu_I v^\alpha \,. \end{aligned} \tag{2.8}$$

But there is a problem with the definition of the affine connection given by Eq. (2.7): the corresponding covariant derivative is not metric compatible.

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^{\beta}{}_{\alpha\mu} g_{\beta\nu} - \Gamma^{\beta}{}_{\alpha\nu} g_{\mu\beta} \\
&= \eta_{IJ} \partial_\alpha (e^I_\mu e^J_\nu) - A^I_{\alpha J} \eta_{IL} e^J_\mu e^L_\nu - A^I_{\alpha J} \eta_{KI} e^J_\nu e^K_\mu \\
&= \eta_{IJ} \partial_\alpha (e^I_\mu e^J_\nu) - A^{IJ}_\alpha e_{J\mu} e_{I\nu} - A^{IJ}_\alpha e_{J\nu} e_{I\mu} \\
&= \eta_{IJ} \partial_\alpha (e^I_\mu e^J_\nu) \,.
\end{aligned} \tag{2.9}$$

Here I have used the antisymmetry of the spin connection in the internal indices. In order to make the covariant derivative metric compatible, the definition of the affine connection needs to be modified as

$$\Gamma^\alpha{}_{\mu\nu} = e^\alpha_I \partial_\mu e^I_\nu + A^I_{\mu J} e^J_\nu e^\alpha_I \,. \tag{2.10}$$

The covariant derivative is to be now understood as

$$\begin{aligned}
\nabla_\mu v^\nu =& \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha \\
=& \partial_\mu v^\nu + v^\alpha e^\nu_I \partial_\mu e^I_\alpha + A^I_{\mu J} e^J_\alpha e^\nu_I v^\alpha \,.
\end{aligned} \tag{2.11}$$

This is a metric-compatible connection, as we see from the following calculation,

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^{\beta}{}_{\alpha\mu} g_{\beta\nu} - \Gamma^{\beta}{}_{\alpha\nu} g_{\mu\beta} \\
&= \eta_{IJ} \partial_\alpha (e^I_\mu e^J_\nu) - \eta_{IJ} e^J_\nu \partial_\alpha e^I_\mu - \eta_{IJ} e^I_\mu \partial_\alpha e^J_\nu - A^I_{\alpha J} \eta_{IL} e^J_\mu e^L_\nu - A^I_{\alpha J} \eta_{KI} e^J_\nu e^K_\mu = 0 \,.
\end{aligned} \tag{2.12}$$

I would like to mention here that the definition of the affine connection given in Eq. (2.10) is same as the so called *tetrad postulate* [17]. In order to see this I need to define a covariant derivative $\bar{\nabla}$ such that its action on a general field P^I_μ is given by

$$\bar{\nabla}_\mu P^I_\nu = \partial_\mu P^I_\nu - \Gamma^\alpha{}_{\mu\nu} P^I_\alpha + A^I_{\mu J} P^J_\nu \,. \tag{2.13}$$

In tetrad postulate the covariant derivative ∇ is considered to be compatible with tetrad i. e.,

$$0 = \nabla_\mu e^I_\nu = \partial_\mu e^I_\nu - \Gamma^\alpha{}_{\mu\nu} e^I_\alpha + A^I_{\mu J} e^J_\nu \,. \tag{2.14}$$

It is analogous to metric compatibility of the Levi-Civita connection $\hat{\nabla}$. Upon contraction with e^{α}_{I}, the above equation becomes same Eq. (2.10). I will however not use the definition of the covariant derivative given by Eq. (2.13) in my calculations.

The important point to note here is that Γ is not necessarily symmetric here. This implies that unlike in General Relativity, we do not have torsion free condition a priori. In terms of the tetrads and the spin connection, the torsion tensor is given by

$$C^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu]} - \Gamma^{\alpha}{}_{\nu\mu]} = e^{\alpha}_{I}\partial_{\mu}e^{I}_{\nu} + A^{I}_{\mu J}e^{J}_{\nu}e^{\alpha}_{I} - e^{\alpha}_{I}\partial_{\nu}e^{I}_{\mu} + A^{I}_{\nu J}e^{J}_{\mu}e^{\alpha}_{I}\,. \tag{2.15}$$

This formalism is thus Palatini formalism where the metric and connection are taken to be independent variables and torsion is determined by on-shell equations. In case of minimally coupled bosonic fields, torsion vanishes on-shell thereby implying the equivalence of vierbein-Einstein-Palatini and metric formalism of General Relativity. So one could ask why at all we need the tetrads and the spin connection we could instead just proceed with a general affine connection independent of the metric. The importance of defining a spin connection is understood when there are fermionic fields in curved spacetime as we will see later.

I will calculate the Riemann tensor corresponding to the affine connection Γ using the usual expression

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\mu\sigma}\,. \tag{2.16}$$

In terms of the tetrads and the spin connection, the Riemann tensor is given by

$$R^{\rho}{}_{\sigma\mu\nu} = F^{I}_{\mu\nu J}e^{\rho}_{I}e^{J}_{\sigma}\,, \tag{2.17}$$

from which the Ricci tensor and Ricci scalar are obtained respectively as

$$R_{\sigma\nu} = F^{I}_{\mu\nu J}e^{\mu}_{I}e^{J}_{\sigma}\,, \tag{2.18}$$

$$R = F^{IJ}_{\mu\nu}e^{\mu}_{I}e^{\nu}_{J}\,. \tag{2.19}$$

The vierbein-Einstein-Palatini action is the Einstein-Hilbert action for gravity where one replaces the Ricci scalar by Eq. (2.19), and the metric determinant by that of tetrads,

$$S[e, A] = \frac{1}{2\kappa} \int_{\mathcal{M}} |e| d^4x \, F^{IJ}_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J} \,, \tag{2.20}$$

where, $\kappa = 8\pi G$. It should be noted that although constructed from the Einstein-Hilbert action, the vierbein-Einstein-Palatini action is not same as the Einstein-Hilbert action. This is because in the Einstein-Hilbert action the only variable is the metric tensor and as a result it is of second order in derivatives. The vierbein-Einstein-Palatini action is first order in derivatives until the connection has been solved and substituted.

This action is first extremised under variations of the vierbein e^{μ}_{I}, keeping $A^{I}_{\mu J}$ fixed. Variation of the determinant gives

$$\delta |e| = -|e| \, e^{I}_{\mu} \delta e^{\mu}_{I} \,. \tag{2.21}$$

Using the antisymmetry of $F^{IJ}_{\mu\nu}$, I can then derive the field equations quite easily,

$$2F^{IJ}_{\lambda\nu} e^{\lambda}_{I} - e^{J}_{\nu} F^{KL}_{\rho\sigma} e^{\rho}_{K} e^{\sigma}_{L} = 0 \,. \tag{2.22}$$

Contracting with $e_{\mu J}$ and using Eq. (2.18) produces the familiar form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \,. \tag{2.23}$$

This equation would be the vacuum Einstein's equation if I could show that ∇ is torsion free, i.e., if $\Gamma^{\alpha}{}_{\mu\nu}$ is symmetric in μ, ν. For this purpose, I vary the action of Eq. (2.20) again, but this time with respect to the spin connection $A^{I}_{\mu J}$, keeping the vierbein fixed. To do this I first simplify the action using the antisymmetry of the spin connection,

$$S[e, A] = \int |e| d^4x \left(\partial_{\mu} A^{IJ}_{\nu} (e^{\mu}_{I} e^{\nu}_{J} - e^{\mu}_{J} e^{\nu}_{I}) + [A_{\mu}, A_{\nu}]^{IJ} e^{\mu}_{I} e^{\nu}_{J} \right) \,. \tag{2.24}$$

Variation with respect to A^{IJ}_{ν} produces the equation

$$\begin{aligned} -e^{\alpha}_{K} \partial_{\mu} e^{K}_{\alpha} (e^{\mu}_{I} e^{\nu}_{J} - e^{\mu}_{J} e^{\nu}_{I}) - e^{\nu}_{J} \partial_{\mu} e^{\mu}_{I} + e^{\nu}_{I} \partial_{\mu} e^{\mu}_{J} - e^{\mu}_{I} \partial_{\mu} e^{\nu}_{J} + e^{\mu}_{J} \partial_{\mu} e^{\nu}_{I} \\ + A^{K}_{\mu I} e^{\mu}_{K} e^{\nu}_{J} - A^{K}_{\mu J} e^{\mu}_{K} e^{\nu}_{I} - A^{K}_{\mu I} e^{\mu}_{J} e^{\nu}_{K} + A^{K}_{\mu J} e^{\mu}_{I} e^{\nu}_{K} = 0 \,. \end{aligned} \tag{2.25}$$

In order to solve this equation for the spin connection, first I contract it with $e_{\nu L}$ and then take trace in LJ. This helps to evaluate the term $A^{K}_{\mu I} e^{\mu}_{K}$ a

$$A^{K}_{\mu I} e^{\mu}_{K} = \partial_{\mu} e^{\mu}_{I} + e^{\mu}_{I} e^{\alpha}_{K} \partial_{\mu} e^{K}_{\alpha} \,. \tag{2.26}$$

It is necessary to evaluate and then to get rid of $A^{K}_{\mu I} e^{\mu}_{K}$ from Eq. (2.25) because, otherwise I cannot remove e^{μ}_{K} from the term and solve for the spin connection. Substituting Eq. (2.26) in Eq. (2.25) gives

$$A_{\mu L I} e^{\mu}_{J} + A_{\mu J L} e^{\mu}_{I} = e_{\nu L} e^{\mu}_{J} \partial_{\mu} e^{\nu}_{I} - e_{\nu L} e^{\mu}_{I} \partial_{\mu} e^{\nu}_{J} \,. \tag{2.27}$$

Cyclic permutation in IJL and then contraction with suitable tetrad produces the following expression for the spin connection.

$$A^{IJ}_{\mu} \equiv \omega^{IJ}_{\mu} = \frac{1}{2} e_{\mu K} \left(\Theta^{KIJ} - \Theta^{IJK} - \Theta^{JKI} \right) \,, \tag{2.28}$$

where for convenience I have defined the quantity Θ^{IJK} as

$$\Theta^{IJK} = e^{I}_{\nu} \left[e^{\mu J} \partial_{\mu} e^{K\nu} - e^{\mu K} \partial_{\mu} e^{\nu J} \right] \,. \tag{2.29}$$

Eq. (2.28) is the the torsion-free spin connection and I have denoted it as ω^{IJ}_{μ}. I can calculate Γ by substituting this expression in Eq. (2.10) and see that

$$\Gamma^{\nu}{}_{\rho\lambda} = \Gamma^{\nu}{}_{\lambda\rho} \equiv \widehat{\Gamma}^{\nu}{}_{\lambda\rho} \,. \tag{2.30}$$

Thus I have recovered the Christoffel symbols on-shell and ∇ can thus be identified as the unique metric-compatible torsion-free connection on the spacetime. Although in usual General Relativity the torsion-free condition is imposed a priori, in the vierbein-Einstein-Palatini formalism only metric-compatibility follows from the definition of Γ; the torsion-free condition comes from the equations of motion. Thus $R_{\mu\nu}$ in Eq. (2.23) can be replaced by $\widehat{R}_{\mu\nu}$, and we see that even though I started with an independent connection, I have recovered General Relativity on-shell. Eq. (2.28) is the expression for the spin connection in the absence of matter, or more precisely in the absence of matter which couples to the spin connection. Thus, in vacuum the vierbein-Einstein-Palatini formalism is equivalent to General Relativity on-shell i. e., upon using the equation of motion of the spin connection.

In the presence of matter the absence of torsion is not guaranteed. I will consider vierbein-Einstein-Palatini action with matter field to see this in more detail,

$$S_{Total} = \frac{1}{2\kappa} \int |e|\, d^4x\, F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J + S_M \,. \tag{2.31}$$

Here the components of the spin connection $A^I_{\mu J}$ are again taken to be independent variables. The equation of motion, obtained by varying this action with respect to the tetrad, is

$$F^{IJ}_{\alpha\mu} e^\alpha_I - \frac{1}{2} e^J_\mu F^{KL}_{\alpha\beta} e^\alpha_K e^\beta_L = \kappa \Theta^J_\mu \,, \tag{2.32}$$

where I have written

$$\Theta^J_\mu = -\frac{\delta S_M}{\delta e^\mu_J} \,. \tag{2.33}$$

After contraction with a suitable tetrad, I get the familiar form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \Theta_{\mu\nu} \,. \tag{2.34}$$

It should be noted that $\Theta_{\mu\nu} = \Theta^J_\mu e_{J\nu}$ is not the usual energy-momentum tensor for the matter, because for $\Theta_{\mu\nu}$ to be energy-momentum tensor, the left hand side of the above equation must be symmetric and torsion-free. If the spin connection couples to matter, the right hand side of Eq. (2.25) will not vanish and in general, the connection will be given as

$$A^{IJ}_\mu = \omega^{IJ}_\mu + \Lambda^{IJ}_\mu \,, \tag{2.35}$$

where Λ^{IJ}_μ corresponds to the contorsion tensor defined in Eq. (1.17) characterises the torsion part of the connection. It should be noted that the Λ term in the equation above comes from the coupling of the spin connection to the matter field considered. As a result this term will always be suppressed by κ. This will be clear once I consider specific fields. I will now discuss how Λ relates to torsion. In terms of Λ, the torsion tensor of Eq. (2.15) can be written as

$$C^\alpha{}_{\mu\nu} = \Lambda^I_{\mu J} e^J_\nu e^\alpha_I - \Lambda^I_{\nu J} e^J_\mu e^\alpha_I \,. \tag{2.36}$$

Using Eq (2.35) and Eq. (1.17), we see that Λ plays the role of contorsion tensor in the vierbein-Einstein-Palatini formalism,

$$S^{\alpha}{}_{\mu\nu} = -\Lambda^{I}_{\mu J} e^{J}_{\nu} e^{\alpha}_{I} \,. \tag{2.37}$$

We will see later that it is the contorsion tensor that couples to spinors, ultimately leading to non-zero torsion [12].

I will discuss how the proper energy-momentum tensor can be obtained on the right hand side of Eq. (2.34). First I need to use the expression of the spin connection given by Eq. (2.35) in Eq. (2.34) and then move all the terms involving Λ to the right hand side. The resulting equation is obtained as

$$\widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R} = \kappa \widehat{T}_{\mu\nu} + \mathcal{O}(\kappa^2) + \cdots \,, \tag{2.38}$$

where quantities with a hat '$\widehat{\ }$' over them are constructed with the torsion-free connection, as before. $\widehat{T}_{\mu\nu}$ is the symmetric and conserved energy momentum tensor and higher order terms are contributions due to dynamically generated torsion. The conservation of the energy-momentum tensor is to be understood in terms of the torsion-free Levi-Civita connection. The procedure for obtaining a symmetric energy-momentum tensor of the spinor field, in particular, was discussed in [12].

I will also discuss the tetrad-only (or simply tetrad) formalism where we have the torsion-free condition a priori. This is thus identical to General Relativity.

2.1 TETRAD FORMULATION OF GENERAL RELATIVITY

In order to recover usual General Relativity from the vierbein-Einstein-Palatini formalism, one needs to work with the torsion-free spin connection ω.

$$(D_{\mu} S)^{I} = \partial_{\mu} S^{I} + \omega^{I}_{\mu J} S^{J} \,, \tag{2.39}$$

where $\omega^{I}_{\mu J}$ is the torsion-free spin connection of Eq. (2.28). To identify the tetrad formulation with General Relativity, the Christoffel symbols of the metric formalism are written using the vierbein and spin connection

$$\widehat{\Gamma}^{\alpha}{}_{\mu\nu} = e^{\alpha}_{I} \partial_{\mu} e^{I}_{\nu} + \omega^{I}_{\mu J} e^{J}_{\nu} e^{\alpha}_{I} \,. \tag{2.40}$$

Metric compatibility of the corresponding Levi-Civita connection helps to express ω in terms of tetrads as given in Eq. (2.28).

As before, I can calculate the Riemann tensor, Ricci tensor and Ricci scalar by successive contraction with the vierbein,

$$\widehat{R}^{\rho}{}_{\sigma\mu\nu} = \widehat{F}^{I}_{\mu\nu J} e^{\rho}_{I} e^{J}_{\sigma} \,, \tag{2.41}$$

$$\widehat{R}_{\sigma\nu} = \widehat{F}^{I}_{\mu\nu J} e^{\mu}_{I} e^{J}_{\sigma} \,, \tag{2.42}$$

$$\widehat{R} = \widehat{F}^{IJ}_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J} \,. \tag{2.43}$$

Here $\widehat{F}^{IJ}_{\mu\nu}$ is the curvature of the connection D. Here Riemann tensor satisfies all the symmetries and identities as it does in General Relativity. The tetrad action for gravity is the Einstein-Hilbert action in which the Ricci scalar has been replaced by Eq. (2.19), and the metric determinant by that of tetrads,

$$S_{tetrad}[e] = \frac{1}{2\kappa} \int |e| d^4x \, \widehat{F}^{IJ}_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J} \,. \tag{2.44}$$

Variation of the action with respect to the tetrads produces the equation

$$2\widehat{F}^{IJ}_{\lambda\nu} e^{\lambda}_{I} - e^{J}_{\nu} \widehat{F}^{KL}_{\rho\sigma} e^{\rho}_{K} e^{\sigma}_{L} = 0 \,. \tag{2.45}$$

Contracting with $e_{\mu J}$, and using Eq. (2.18), I get the familiar form

$$\widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R} = 0 \,. \tag{2.46}$$

If I include matter fields, the tetrad action reads

$$S_{Total} = \frac{1}{2\kappa} \int |e| \, d^4x \, \widehat{F}^{IJ}_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J} + S_M \,, \tag{2.47}$$

where $S_M = \int |e| \, d^4x \, \mathcal{L}_M$ is the action for any matter field present. The equation of motion obtained by variation with respect to the tetrad is thus

$$\widehat{F}^{IJ}_{\alpha\mu} e^{\alpha}_{I} - \frac{1}{2} e^{J}_{\mu} \widehat{F}^{KL}_{\alpha\beta} e^{\alpha}_{K} e^{\beta}_{L} = \kappa \widehat{T}_{\mu\alpha} e^{\alpha J} \,, \tag{2.48}$$

where $\widehat{T}_{\mu\alpha}$ is the usual energy-momentum tensor for the matter. As before, I can contract this equation with the tetrad to obtain the familiar form, $\widehat{G}_{\mu\nu} = \kappa \widehat{T}_{\mu\nu}$.

3

MATTER FIELDS

In this chapter, I will consider different matter fields in the vierbein-Einstein-Palatini formalism. I will demonstrate that for bosonic fields, torsion either vanishes on-shell or it is taken to be zero a priori for the sake of gauge symmetry. In such cases one can proceed with a torsion-free tetrad formalism identical to General Relativity. But we will see that in case of fermionic field, torsion has effects on the Einstein's equation as well as the Dirac equation.

3.1 SCALAR FIELD

The total action with massless real scalar field is given by

$$S = S[e, A] - \frac{1}{2} \int |e| d^4x \nabla_\mu \phi \nabla^\mu \phi \,, \tag{3.1}$$

where $S[e, A]$ is the vierbein-Einstein-Palatini action of Eq. (2.20). It should be noted that although I have written the derivative of the scalar field with ∇_μ, it is essentially the partial derivative ∂_μ. In other words it should not matter whether I take the derivative with torsion-free $\widehat{\nabla}_\mu$, torsionful ∇_μ or ordinary partial derivative. The equation obtained by extremising the action with the scalar field is

$$\nabla_\mu \nabla^\mu \phi = 0 \,. \tag{3.2}$$

Although the above equation contains the torsionful connection it is essentially the same as the equation with the torsion-free connection. I can break the kinetic term in the following way

$$\nabla_\mu \nabla^\mu \phi = \widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - S^\mu{}_{\alpha\mu} \nabla^\alpha \phi \,. \tag{3.3}$$

Recalling the expression for the contorsion tensor from Eq. (1.17), we see that the last term in the above expression vanishes because of the antisymmetry of the contorsion tensor in the first and last indices. Thus I can neglect torsion if I am dealing with minimally coupled scalar field alone. I will consider an interesting example of non-minimal scalar field in chapter 4 and see how it can potentially couple to torsion.

I will also extremise the action with respect to the tetrads to find out Einstein's equation. The equation is obtained as

$$2F^{IJ}_{\mu\nu} e^\mu_I - e^J_\nu F^{KL}_{\alpha\beta} = 2\kappa (e^{J\mu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} e^J_\nu \nabla_\alpha \phi \nabla^\alpha \phi) \,. \tag{3.4}$$

Upon contraction with suitable tetrad, the above equation can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) \,, \tag{3.5}$$

where I have used the definition of Ricci tensor and Ricci scalar from Eq. (2.18) and Eq. (2.19) respectively. Also, because the scalar Lagrangian does not contain the spin connection, variation of the total action with respect to it gives the torsion-free expression of the connection. I can thus identify the above equation with Einstein's equation in General Relativity with scalar field.

3.2 ELECTROMAGNETIC FIELD

Usually in the case Maxwell Lagrangian a torsionful connection is not considered and consequently there is no equation of motion for torsion. This is because a torsionful derivative breaks the gauge symmetry of electromagnetic field. I will consider the total action with em field to demonstrate this.

$$S = \frac{1}{2\kappa} \int |e| d^4x F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J - \frac{1}{4} \int e d^4x F_{\mu\nu} F^{\mu\nu}. \tag{3.6}$$

Here,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{3.7}$$

Usually in differential form notation F is defined as $F = dA$. In case of torsion-free connection, dA can be written as in Eq. (3.7). But here ∇ is the connection that has torsion. Writing the affine connection as

$$\Gamma^{\alpha}{}_{\mu\nu} = \widehat{\Gamma}^{\alpha}{}_{\mu\nu} - S^{\alpha}{}_{\mu\nu}, \tag{3.8}$$

I get

$$F_{\mu\nu} = \widehat{\nabla}_{\mu} A_{\nu} - \widehat{\nabla}_{\nu} A_{\mu} - C^{\alpha}{}_{\mu\nu} A_{\alpha}, \tag{3.9}$$

with $S^{\alpha}{}_{\mu\nu}$ being the contorsion tensor. In the above expression of the field tensor only the antisymmetric part of contorsion appears because the symmetric part cancels out. I will denote the torsion-free part of the field tensor as

$$\widehat{F}_{\mu\nu} = \widehat{\nabla}_{\mu} A_{\nu} - \widehat{\nabla}_{\nu} A_{\mu}, \tag{3.10}$$

which is same as $\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. The total action now becomes

$$\begin{aligned} S = \frac{1}{2\kappa} \int |e| d^4x F^{IJ}_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J} - \frac{1}{4} \int |e| d^4x \widehat{F}_{\mu\nu} \widehat{F}^{\mu\nu} + \frac{1}{2} \int |e| d^4x \widehat{F}^{\mu\nu} C^{\alpha}{}_{\mu\nu} A_{\alpha} \\ - \frac{1}{4} \int |e| d^4x C^{\alpha}{}_{\mu\nu} C^{\beta\mu\nu} A_{\alpha} A_{\beta}. \end{aligned} \tag{3.11}$$

The point to note here is that the Lagrangian of electromagnetic field in the above action is not gauge invariant because of the last two terms. Thus torsion breaks gauge symmetry of the field.

It is apparent from the above that for the case of bosonic fields, torsion either does not have any effect on the field equation or it is not considered for the sake of preserving gauge symmetry. In these cases I can assume the torsion-free condition a priori where the spin connection is written completely in terms of the tetrads similar to the Christoffel symbols in General Relativity. I am thus working in the tetrad formulation of General Relativity as discussed in chapter 1.

3.3 FERMIONIC FIELD

For fermions, I will consider both tetrad formulation and vierbein-Einstein-Palatini formalism. First I will discuss fermionic fields in the torsion-free tetrad formulation.

Next I will consider the vierbein-Einstein-Palatini formalism and compare the equations with those in the tetrad formulation. We will see how inclusion of torsion gives rise to extra terms in the equations.

3.3.1 *Fermionic field in tetrad formalism*

The advantage of having the spin connection is that I can write an action for fermionic fields in curved spacetime. The γ-matrices are defined on the flat internal space and then brought to the spacetime using tetrads, while the covariant derivative on the fermionic field is defined in terms of the spin connection. In general, the spin connection is treated as an independent variable while considering the fermionic field [10–12, 18]. I will discuss this in Sec. 4.2. When I restrict to the torsion-free case however, the connection is not a free variable, but ω_μ^{IJ} of Eq. (2.28). The total action of gravity with a minimally coupled fermion in this case is written as [18]

$$S[e, A, \psi] = \frac{1}{2\kappa} S_{tetrad}[e] + \int |e| d^4x \left[\frac{i}{2} \left(\bar{\psi}\gamma^K e_K^\mu \, {}^\psi\widehat{D}_\mu \psi - (\bar{\psi}\gamma^K e_K^\mu \, {}^\psi\widehat{D}_\mu \psi)^\dagger \right) + im\bar{\psi}\psi \right] , \tag{3.12}$$

where $S_{tetrad}[e]$ is the gravity action given in Eq. (2.44). The covariant derivative ${}^\psi\widehat{D}_\mu$ acts on the spinor ψ as

$${}^\psi\widehat{D}_\mu \psi = \partial_\mu \psi - \frac{i}{4} \omega_\mu^{IJ} \sigma_{IJ} \psi \, , \tag{3.13}$$

where $\sigma_{IJ} = \frac{i}{2} [\gamma_I \, , \gamma_J]$.

I will calculate the second term in the fermionic Lagrangian

$$\begin{aligned}
&(\bar{\psi}\gamma^K e_K^\mu \, {}^\psi\widehat{D}_\mu \psi)^\dagger \\
&= (\bar{\psi}\gamma^K \partial_\mu \psi)^\dagger e_K^\mu + \frac{i}{4} \omega_\mu^{IJ} e^{\mu K} (\bar{\psi}\gamma_K \sigma_{IJ} \psi)^\dagger \\
&= \partial_\mu \psi^\dagger \gamma^{K\dagger} \gamma_0^\dagger \psi + \frac{i}{4} \omega_\mu^{IJ} e^{\mu K} \psi^\dagger \sigma_{IJ}^\dagger \gamma_K^\dagger \gamma_0^\dagger \psi \\
&= \partial_\mu \bar{\psi} \gamma^K e_K^\mu \psi + \frac{i}{4} \omega_\mu^{IJ} e^{\mu K} \bar{\psi} \sigma_{IJ} \gamma_K \psi \, .
\end{aligned} \tag{3.14}$$

Here I have used the properties of γ and σ-matrices from Appendix D. The fermionic Lagrangian can thus be written as

$$\mathcal{L}_F = \frac{i}{2}\left(\bar{\psi}\gamma^K e^\mu_K \partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^K e^\mu_K\psi - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\right) + im\bar{\psi}\psi\,. \tag{3.15}$$

The γ and σ matrices carry internal flat space indices and have the properties for metric signature $(-+++)$. It should be noted here that for the choice of signature $(+---)$, which is popular in quantum field theory, γ needs to replace by $-i\gamma$ in all of these expressions (see Appendix D).

Extremising the action of Eq. (3.12) with respect to the tetrad and the fermion, the equations of motion are obtained as

$$\delta e^\nu_J : \widehat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\widehat{R} = \kappa\widehat{T}_{\mu\nu}(\psi, \bar{\psi})\,, \tag{3.16a}$$

$$\delta\bar{\psi} : 2\gamma^K e^\mu_K\partial_\mu\psi + e^\alpha_I\partial_\mu e^I_\alpha\, \gamma^K e^\mu_K\psi + \partial_\mu e^\mu_K\gamma^K\psi - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\{\gamma_K, \sigma_{IJ}\}\psi + m\psi = 0\,. \tag{3.16b}$$

In addition, by varying ψ I get an equation which is the adjoint of Eq. (3.16b). Here $\widehat{T}_{\mu\nu}(\psi, \bar{\psi})$ is the symmetric and conserved energy-momentum tensor of the fermionic field,

$$\widehat{T}_{\mu\nu}(\psi, \bar{\psi}) = \frac{i}{4}\left[(\partial_\mu\bar{\psi})\gamma_I\psi e^I_\nu - \bar{\psi}\gamma_I(\partial_\mu\psi)e^I_\nu + \frac{i}{4}\omega^{IJ}_\mu e^K_\nu\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi + (\mu\leftrightarrow\nu)\right] + img_{\mu\nu}\bar{\psi}\psi\,. \tag{3.17}$$

It should be noted that for obtaining the above expression for $\widehat{T}_{\mu\nu}$, I have also varied the spin connection ω^{IJ}_μ with respect to tetrads. In fact the terms that come from the variation of the spin connection, along with Eq. (3.16b), give the symmetric form of $\widehat{T}_{\mu\nu}$. Eq. (3.16b) is the Dirac equation in torsion-free curved spacetime. I can cast the Dirac equation in a familiar form by using the expression for ω^{IJ}_μ of Eq. (2.28),

$$\gamma^K e^\mu_K\, {}^\psi\widehat{D}_\mu\psi + m\psi = 0\,. \tag{3.18}$$

3.3.2 *Fermionic field in vierbein-Einstein-Palatini formalism*

In the tetrad formulation we saw how one can write fermions in gravity or more specifically torsion-free gravity. In the vierbein-Einstein-Palatini formalism fermionic field becomes more interesting because of its ability to couple to torsion. I will consider the action to see this in detail.

$$S[e, A, \phi, \psi] = S[e, A] + \int |e|\, d^4x \left[\frac{i}{2}\left(\bar{\psi}\gamma^K e^\mu_K\, {}^\psi D_\mu\psi - (\bar{\psi}\gamma^K e^\mu_K\, {}^\psi D_\mu\psi)^\dagger\right) + im\bar{\psi}\psi\right], \tag{3.19}$$

where, the derivative ${}^\psi D$ acts on the spinor ψ via the spin connection A,

$${}^\psi D_\mu\psi = \partial_\mu\psi - \frac{i}{4}A^{IJ}_\mu\sigma_{IJ}\psi\,. \tag{3.20}$$

As before, the fermionic Lagrangian can be written in the following manner,

$$\mathcal{L}_F = \frac{i}{2}\left(\bar{\psi}\gamma^K e^\mu_K\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^K e^\mu_K\psi - \frac{i}{4}A^{IJ}_\mu e^{\mu K}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\right) + im\bar{\psi}\psi\,. \tag{3.21}$$

Extremising the action with respect to the different variables produces the following equations.

$$\delta e^\nu_J \;:\; R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{i\kappa}{2}\left[(\partial_\nu\bar{\psi})\,\gamma_I\psi e^I_\mu - \bar{\psi}\gamma_I(\partial_\nu\psi)e^I_\mu + \frac{i}{4}A^{IJ}_\nu e^K_\mu\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\right] + i\kappa m g_{\mu\nu}\bar{\psi}\psi\,, \tag{3.22a}$$

$$\delta A^{IJ}_\nu \;:\; A^{IJ}_\mu = \omega^{IJ}_\mu[e] + \frac{\kappa}{8}\bar{\psi}\{\gamma_K, \sigma^{IJ}\}\psi e^K_\mu\,, \tag{3.22b}$$

$$\delta\bar{\psi} \;:\; 2\gamma^K e^\mu_K\partial_\mu\psi + e^\alpha_I\partial_\mu e^I_\alpha\;\gamma^K e^\mu_K\psi + \partial_\mu e^\mu_K\gamma^K\psi - \frac{i}{4}A^{IJ}_\mu e^{\mu K}\{\gamma_K, \sigma_{IJ}\}\psi + m\psi = 0\,. \tag{3.22c}$$

I can simplify Eq. (3.22c) by using identities of γ and σ matrices; and the definition of torsion in Eq. (2.15),

$$\gamma^K e^\mu_K\partial_\mu\psi + \frac{1}{2}C^\alpha{}_{\mu\alpha}e^\mu_K\gamma^K\psi - \frac{i}{4}A^{IJ}_\mu e^{\mu K}\gamma_K\sigma_{IJ}\psi + m\psi = 0. \tag{3.23}$$

Let us now see what happens to the above equation if I use the on-shell expression of the spin connection given by Eq. (3.22b). The first term in this expression is exactly the same as in Eq. (2.28). The last term has appeared due to the fermionic field. This term can be identified with the contorsion Λ_μ^{IJ} which was defined in Eq. (2.35),

$$\Lambda_\mu^{IJ} = \frac{\kappa}{8}\bar{\psi}\{\gamma_K, \sigma^{IJ}\}\psi e_\mu^K . \tag{3.24}$$

I can use the identity

$$\{\sigma^{IJ}, \gamma^K\} = 2\epsilon^{IJKL}\gamma_L\gamma_5 , \tag{3.25}$$

to write Λ_μ^{IJ} as

$$\Lambda_\mu^{IJ} = \frac{\kappa}{4}\epsilon^{IJKL}\bar{\psi}\gamma_L\gamma_5\psi e_{K\mu} . \tag{3.26}$$

This term results in the following non-vanishing expression for the on-shell torsion tensor:

$$^{OS}C^\alpha{}_{\mu\nu} = \frac{\kappa}{2}\epsilon^{IJKL}\bar{\psi}\gamma_L\gamma_5\psi e_I^\alpha e_{J\mu} e_{K\nu} . \tag{3.27}$$

Clearly, on-shell torsion, generated by a fermion source, is totally antisymmetric. If I identify the torsion tensor in Eq. (3.23) with on-shell torsion tensor, we can see that the second term in the equation goes away due to the total antisymmetry of the torsion tensor given by Eq. (3.27). Thus I am left with the equation

$$\gamma^K e_K^\mu \partial_\mu\psi - \frac{i}{4}A_\mu^{IJ} e^{\mu K}\gamma_K\sigma_{IJ}\psi + m\psi = 0 \tag{3.28}$$

or simply

$$\gamma^K e_K^\mu \; {}^\psi D_\mu \psi + m\psi = 0 . \tag{3.29}$$

This looks like the equation I obtained for fermionic field in tetrad formulation, but it is different in the sense that the spin connection, and thus the derivative, now contain non-zero torsion. If I now use Eq. (3.22b) for the on-shell expression of the spin connection, I get a nonlinear spinor equation with cubic term resulting from torsion, as has been noted in [12, 19],

$$\gamma^K e_K^\mu \partial_\mu\psi - \frac{i}{4}\omega_\mu^{IJ} e^{\mu K}\gamma_K\sigma_{IJ}\psi + m\psi - \frac{i\kappa}{64}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\{\gamma^K, \sigma^{IJ}\}\psi = 0 . \tag{3.30}$$

I will also use the on-shell expression of the spin connection given by Eq. (3.22b), and the Dirac equation of Eq. (3.22c) in Eq. (3.22a). After some lengthy but straightforward calculations I get the following equation.

$$\widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R} = \kappa \widehat{T}_{\mu\nu}(\psi, \bar{\psi}) - \frac{3\kappa^2}{16} g_{\mu\nu} \bar{\psi} \gamma_I \gamma_5 \psi \bar{\psi} \gamma^I \gamma_5 \psi \,. \tag{3.31}$$

Here, $\widehat{T}_{\mu\nu}(\psi, \bar{\psi})$ is the symmetric and conserved energy-momentum that comes from the torsion-free matter Lagrangian as obtained in Eq. (3.17), which comes from a torsion-free theory of gravity. I should mention how I have obtained this symmetric energy-momentum tensor. As we can see from Eq. (3.22a), in contrast to tetrad formalism, the right hand side does not contain the symmetric energy-momentum tensor. This is because the left hand side contains torsion components and is not symmetric either. The idea is to use the on-shell expression of Λ from Eq. (3.24) in the Einstein's equation. There are two ways I can proceed to obtain the symmetric energy-momentum. The first way is to directly use the expression for Λ in the quantities on the left hand side of Eq. (3.22a). I find that the Ricci scalar part is not affected by this but the Ricci tensor breaks into symmetric and antisymmetric parts as

$$R_{\mu\nu} = \widehat{R}_{\mu\nu} + \frac{1}{2}(\Theta_{\mu\nu} - \Theta_{\nu\mu}) \,, \tag{3.32}$$

where I have denoted the $\mathcal{O}(\kappa)$ terms of the right hand side of Eq. (3.22a) as $\Theta_{\mu\nu}$. The antisymmetric part above, when taken to the right hand side, gives the symmetric energy-momentum tensor. The second way is to find the symmetric and antisymmetric components of Eq. (3.22a). In this case I can find the antisymmetric component directly. For the symmetric component of $R_{\mu\nu}$, I can show that it is the same as the Ricci tensor calculated from the torsion-free spin connection ω.

The conservation of $T_{\mu\nu}(\psi, \bar{\psi})$ is to be understood in terms of the torsion-free derivative operator $\widehat{\nabla}$. The additional term of $\mathcal{O}(\kappa^2)$ has appeared due to torsion. Using generalised Fierz identities for the spinor field [20–22], I can write Einstein's equations as

$$\widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R} = \kappa \widehat{T}_{\mu\nu}(\psi, \bar{\psi}) - g_{\mu\nu} \frac{3\kappa^2}{16} \left((\bar{\psi}\gamma_5\psi)^2 - (\bar{\psi}\psi)^2 \right) \,. \tag{3.33}$$

I will conclude this section with a discussion on the nonlinear Dirac Eq. (3.30). As already mentioned, the effect of torsion on the equation is realised through the cubic term. I can write an effective Lagrangian without torsion that gives me the same equation as obtained above. To do this I need to modify the torsion-free spinor Lagrangian of (3.15) with the addition of a quartic term as

$$\mathcal{L}_F = \frac{i}{2}\left(\bar{\psi}\gamma^K e^\mu_K \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^K e^\mu_K \psi - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi \right. \\ \left. - \frac{i\kappa}{64}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\bar{\psi}\{\gamma^K, \sigma^{IJ}\}\psi\right) + im\bar{\psi}\psi\,. \tag{3.34}$$

I can write the above Lagrangian in a better form using Fierz identities as

$$\mathcal{L}_F = \frac{i}{2}\left(\bar{\psi}\gamma^K e^\mu_K \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^K e^\mu_K \psi - \frac{i}{2}\epsilon_{IJKL}\omega^{IJ}_\mu e^{\mu K}\bar{\psi}\gamma^L\gamma_5\psi \right. \\ \left. - \frac{3i\kappa}{8}\left((\bar{\psi}\gamma_5\psi)^2 - (\bar{\psi}\psi)^2\right)\right) + im\bar{\psi}\psi\,. \tag{3.35}$$

This Lagrangian also produces the $\mathcal{O}(\kappa^2)$ term appearing in the Einstein's equation. Also it gives the same nonlinear Dirac equation of (3.30).

3.3.3 *Conservation of spinor current*

In order to find the current conservation equation, first I need to obtain the equation of motion of $\bar{\psi}$. There are two ways to obtain the equation. I can either extremise the action with respect to ψ or take adjoint of the ψ-equation (3.28). Either way, the equation is obtained as

$$\partial_\mu \bar{\psi}\gamma^K e^\mu_K + \frac{i}{4}A^{IJ}_\mu e^{\mu K}\bar{\psi}\sigma_{IJ}\gamma_K - m\bar{\psi} = 0\,. \tag{3.36}$$

Now following the usual procedure of pre-multiplying Eq. (3.28) with $\bar{\psi}$, post-multiplying Eq. (3.36) with ψ and adding them together gives

$$\partial_\mu(\bar{\psi}\gamma^K\psi) + \frac{i}{4}A^{IJ}_\mu e^{\mu K}\bar{\psi}[\sigma_{IJ}, \gamma_K]\psi = 0\,. \tag{3.37}$$

Now, using the expression for the commutator from Eq. (D.10) the above equation can be written as

$$
\begin{aligned}
& e^\mu_K \partial_\mu (\bar{\psi}\gamma^K \psi) + \frac{i}{4} A^{IJ}_\mu e^{\mu K} \bar{\psi} 2i (\eta_{JK}\gamma_I - \eta_{IK}\gamma_J)\psi = 0 \\
\Rightarrow \quad & e^\mu_K \partial_\mu (\bar{\psi}\gamma^K \psi) + A^{IJ}_\mu e^\mu_I \bar{\psi}\gamma_J \psi = 0 \\
\Rightarrow \quad & \partial_\mu (\bar{\psi}\gamma^K e^\mu_K \psi) - \bar{\psi}\gamma^K \psi \partial_\mu e^\mu_K + A^{IJ}_\mu e^\mu_I \bar{\psi}\gamma_J \psi = 0 \\
\Rightarrow \quad & \partial_\mu (\bar{\psi}\gamma^K e^\mu_K \psi) - \bar{\psi}\gamma^L \psi e^\nu_L e^K_\nu \partial_\mu e^\mu_K + A^{IJ}_\mu e^\mu_I e_{J\nu} e^{\nu K} \bar{\psi}\gamma_K \psi = 0 \\
\Rightarrow \quad & \partial_\mu J^\mu + (e^\mu_I \partial_\mu e^I_\nu + A^{IJ}_\mu e^\mu_I e_{J\nu}) J^\nu = 0 \\
\Rightarrow \quad & \nabla_\mu J^\mu = 0\,,
\end{aligned}
\tag{3.38}
$$

where, $J^\mu = e^\mu_I \bar{\psi}\gamma^I \psi$ is the spinor current in curved spacetime. In writing the last step, I have used the definition of Γ from Eq. (2.10). Although I have written the equation with torsionful derivative ∇, it is same as the torsion-free derivative $\widehat{\nabla}$. The reason for this is that it is a divergence equation where the effect of torsion vanishes by the same argument given in (3.1).

4

CONFORMAL TRANSFORMATION

In this chapter[1] I will investigate conformal transformations of the vierbeins and the spin connection. The motivation for this investigation is twofold. First, if the vierbein-Einstein-Palatini action is exactly equivalent to the second-order Einstein-Hilbert action of gravity, all matter fields should couple to gravity ‘in the same way’ in both formulations. More precisely, the corresponding stress-energy tensors for matter should be equivalent in the two formulations, and matter fields ought to transform in the same way in both the formulations. While this is a trivial issue for minimally coupled matter fields, it turns out that for non-minimally coupled fields, such as the conformally coupled scalar which I investigate here, the field equations behave differently under conformal transformations in the two formulations. The other motivation is to study the conformal transformation of spin connection, which is useful in studying the conformal properties of fermions propagating on a curved background. Since the spin connection has torsion components which couple to fermions, its behaviour under conformal transformations affects that of fermions.

I will consider different possibilities of how torsion is affected by conformal transformations. First I will discuss Nieh-Yan theory [23], in which torsion was considered to “play the role of gauge potential for the conformal transformation group." However, when torsion is taken to zero, this theory does not reduce to pure Einstein gravity, i.e., General Relativity based on pure Riemannian geometry. Next I will discuss a theory where torsion remains invariant under conformal transformations. I will show that Nieh-Yan theory and the one with invariant torsion correspond to two limits of a general transformation of the spin connection which interpolates between these two limits. I will also consider dynamically generated or on-shell torsion which is the expression for torsion obtained by solving the equation of motion. I will show that torsion, dy-

1 The work reported here is based on the paper "Different types of torsion and their effect on the dynamics of fields", Subhasish Chakrabarty and Amitabha Lahiri, Eur. Phys. J. Plus 133, 6, 242 (2018).

namically generated by the Dirac field as in [12, 24], transforms homogeneously under conformal transformation. In other words, unlike in Nieh-Yan theory, on-shell torsion does not have any inhomogeneous conformal transformation. I will also discuss the possibility of on-shell torsion being generated by a conformal scalar field. However, for the scalar field, I will show that on-shell torsion indeed transforms inhomogeneously.

Conformal transformations were introduced by Weyl in an attempt to unify electromagnetism and general relativity [25], and have been useful in studying various properties of curved spacetimes [26]. Conformal transformations have been widely used in studying asymptotic flatness and initial value problem [27–31], propagation of massless fields on a gravitational background [32–40] and exact solutions [41–47]. Conformal invariance is required where scale-independence is fundamental to our understanding of the system. Conformal invariance is also important in the study of quantum field theory on curved spacetime [48–51]. It has been suggested that cosmology based on conformal gravity, or more specifically based on the Weyl tensor, can provide alternatives to the usual cosmologies with dark matter and cosmological constant [52, 53].

A conformal transformation is the scaling of the spacetime metric $g_{\mu\nu}$ with a strictly positive, smooth function Ω^2,

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \,. \tag{4.1}$$

In this nomenclature and related notational conventions, I have followed [27]. I should mention that some authors call this a Weyl transformation, reserving the name 'conformal transformations' for what are called 'conformal isometries' in [27] (for a discussion on the nomenclature, see [54]). A conformal isometry ϕ on a manifold M is a diffeomorphism $\phi : M \to M$ such that its action on the metric is given by $\phi^* g_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

Conformal transformations alter lengths of spacetime intervals, but preserve angles. The conformally transformed spacetime and the original one have the same causal structure. Since Ω is a function of spacetime, the transformation of metric affects different entities like the Christoffel symbols, Riemann tensor and hence the Einstein-Hilbert action. For gauge fields in four dimensions, the matter action remains invariant under conformal transformation, while for other kinds of matter fields like the scalar, the action needs to be modified. Conformal transformation of the metric transforms the Christoffel symbols as

$$\widehat{\Gamma}^{\alpha}{}_{\mu\nu} \to \widehat{\Gamma}^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{(\mu}\widehat{\nabla}_{\nu)} \ln\Omega - g_{\mu\nu}g^{\alpha\beta}\widehat{\nabla}_{\beta} \ln\Omega\,, \tag{4.2}$$

where the symmetric combination is defined as $A_{(\alpha}B_{\beta)} = A_\alpha B_\beta + A_\beta B_\alpha$. The quantities which are defined using the torsion-free connection will be denoted with a hat '$\widehat{\ }$' as in previous chapters. The transformation of torsion-free Ricci scalar can be written as

$$\begin{aligned}\widehat{R} \to \Omega^{-2}\{&\widehat{R} - 2(n-1)g^{\mu\nu}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu} \ln\Omega \\ &- (n-1)(n-2)(\widehat{\nabla}_{\mu} \ln\Omega)(\widehat{\nabla}^{\mu} \ln\Omega)\}\,.\end{aligned} \tag{4.3}$$

This is the general formula in n spacetime dimensions. In this chapter I will be concerned with the case where $n = 4$.

The equation of motion of massless scalar field,

$$\widehat{\nabla}_{\mu}\widehat{\nabla}^{\mu}\phi = 0\,, \tag{4.4}$$

is not invariant under conformal transformation. The remedy is to modify the equation with the addition of a non-minimal term,

$$\widehat{\nabla}_{\mu}\widehat{\nabla}^{\mu}\phi - \frac{1}{6}\widehat{R}\phi = 0\,. \tag{4.5}$$

The above equation can be obtained from the total action

$$S(\phi, g) = S_{EH}[g] - \int \sqrt{-g}\, d^4x \left[\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{12}\widehat{R}\phi^2\right]\,, \tag{4.6}$$

where, $S_{EH}[g]$ is the Einstein-Hilbert action. This matter part of the action is invariant under the conformal transformation of Eq. (4.1) provided the scalar field transforms as

$$\phi \to \Omega^{-1}\phi\,. \tag{4.7}$$

Variation of the action with respect to the metric produces the energy-momentum tensor corresponding to the conformal scalar field, which now includes a part that depends on the geometry because of the $\widehat{R}\phi^2$ term,

$$\widehat{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}\widehat{G}_{\mu\nu}\phi^2 + \frac{1}{6}\left[g_{\mu\nu}\widehat{\nabla}_\sigma\widehat{\nabla}^\sigma\phi^2 - \widehat{\nabla}_\mu\widehat{\nabla}_\nu\phi^2\right]. \quad (4.8)$$

This $\widehat{T}_{\mu\nu}$ is a conserved tensor as expected,

$$\widehat{\nabla}^\mu\widehat{T}_{\mu\nu} = 0\,. \quad (4.9)$$

In this chapter I will discuss the conformal transformation of the vierbein and the spin connection and investigate the conformal properties of the action and equations in the vierbein-Einstein-Palatini formalism. In Sec. 4.1, I will go through the basics of conformal transformation, conformally invariant massless scalar field and fermionic field in the tetrad formulation of General Relativity. I will consider massless fields because a mass term usually breaks conformal symmetry. In Sec. 4.2, I will investigate the conformal transformation of the vierbein and the spin connection. The spin connection is an independent variable at the level of action and thus its transformation remains indeterminate at this stage. It is however possible to make different choices of transformations without disturbing metric compatibility. In this respect I will discuss two such choices: one with inhomogeneously transforming torsion (Nieh-Yan theory) and other with invariant torsion which does not seem to have been discussed in literature before. In Sec. 4.3 I will consider the conformal properties of dynamically generated torsion with specific fields. I will write a general transformation of the off-shell spin connection which, in suitable limits, reduces to Nieh-Yan theory or invariant torsion.

4.1 CONFORMAL TRANSFORMATION IN TETRAD FORMULATION

In this section I will consider the conformal transformation of different entities in the torsion-free tetrad formulation. Because this formalism is equivalent to General Relativity, we can expect that fields and their actions will transform under conformal transformations in the same way in the tetrad formalism as they do in the usual metric formulation of General Relativity.

The transformation of $g_{\mu\nu}$ suggests that the tetrads should transform in the following manner,

$$e^I_\mu \to \Omega e^I_\mu \,, \tag{4.10}$$

while the co-tetrads should transform as

$$e^\mu_I \to \Omega^{-1} e^\mu_I \,. \tag{4.11}$$

The conformal transformation of the spin connection, given by ω^{IJ}_μ of Eq. (2.28), can be found from the transformation of the vierbein alone,

$$\omega^I_{\mu J} \to \omega^I_{\mu J} + (e^I_\mu e^\nu_J - e_{\mu J} e^{\nu I})\partial_\nu \ln \Omega \,. \tag{4.12}$$

It was argued in [55] that the above equation is the conformal transformation of the spin connection even in the presence of fermionic matter. We will however see in Sec. 4.3.2 that when the spin connection is treated as an independent variable, the above transformation may not be quite correct.

Eq. (4.12) leads to the following transformation of the Ricci scalar

$$\widehat{F}^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \to \Omega^{-2} \left[\widehat{F}^{IJ}_{\mu\nu} e^\mu_I e^\nu_J - 6\widehat{\nabla}_\mu \widehat{\nabla}^\mu \ln \Omega - 6 \left(\widehat{\nabla}_\mu \ln \Omega \right) \left(\widehat{\nabla}^\mu \ln \Omega \right) \right] . \tag{4.13}$$

The covariant derivative here is to be understood as being written in terms of ω ,

$$\widehat{\nabla}_\mu V_\nu = \partial_\mu V_\nu - e^\alpha_I \partial_\mu e^I_\nu V_\alpha - \omega^I_{\mu J} e^J_\nu e^\alpha_I V_\alpha \,. \tag{4.14}$$

This is the same torsion-free covariant derivative corresponding to the Christoffel symbols written in a different form.

4.1.1 *Conformal scalar in tetrad formulation*

I will consider the conformal scalar field in tetrad formulation in this section. In terms of tetrads the action of Eq. (4.6) can be written as

$$S[e, A, \phi] = S_{tetrad}[e, A] + \int |e| d^4x \left[-\frac{1}{2} e^\mu_I e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} \widehat{F}^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2 \right] . \tag{4.15}$$

Extremising the action with respect to the independent variables produces the following equations of motion.

$$\delta e^{\nu}_{J} : \quad \widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R} = \kappa \widehat{T}_{\mu\nu} \, , \tag{4.16a}$$

$$\delta \phi : \quad \widehat{\nabla}_{\mu} \widehat{\nabla}^{\mu} \phi - \frac{1}{6} \widehat{R} \phi = 0 \, . \tag{4.16b}$$

Here I have contracted the equations with suitable tetrads to cast them in familiar forms as in General Relativity. $T_{\mu\nu}$ is the same conserved energy momentum tensor obtained in Eq. (4.8) using the metric formulation of General Relativity. The scalar field equation above is invariant under the conformal transformation of tetrads and ϕ. These results are expected as the tetrad formulation is nothing but General Relativity with different variables.

4.1.2 *Conformal invariance of fermionic field in tetrad formulation*

Let us recall the fermionic Lagrangian given in Eq. (3.15).

$$\mathcal{L}_F = \frac{i}{2} \left(\bar{\psi} \gamma^K e^{\mu}_{K} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^K e^{\mu}_{K} \psi - \frac{i}{4} \omega^{IJ}_{\mu} e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \psi \right) \, . \tag{4.17}$$

The above Lagrangian transforms homogeneously with conformal weight of -4 under conformal transformation of tetrads, provided the fermionic field transforms as

$$(\psi, \, \bar{\psi}) \rightarrow \left(\Omega^{-\frac{3}{2}} \psi, \, \Omega^{-\frac{3}{2}} \bar{\psi} \right) \, . \tag{4.18}$$

The covariance of the Lagrangian can be seen by noting that the inhomogeneous transformations of the first two terms cancel out and for the last term involving connection,

$$\begin{aligned}
\omega^{IJ}_{\mu} e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \psi \quad \rightarrow \quad & \Omega^{-4} \omega^{IJ}_{\mu} e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \\
& + \Omega^{-4} \left(e^{I}_{\mu} e^{\nu J} - e^{J}_{\mu} e^{\nu I} \right) \partial_{\nu} \ln \Omega \; e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \psi \\
= \quad & \Omega^{-4} \omega^{IJ}_{\mu} e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \\
& + \Omega^{-4} \left(\eta^{IK} e^{\nu J} - \eta^{JK} e^{\nu I} \right) \partial_{\nu} \ln \Omega \; \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \psi \\
= \quad & \Omega^{-4} \omega^{IJ}_{\mu} e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \\
& + 2 \, \Omega^{-4} e^{\nu J} \partial_{\nu} \ln \Omega \; \bar{\psi} \{ \gamma^I , \sigma_{IJ} \} \psi \\
= \quad & \Omega^{-4} \omega^{IJ}_{\mu} e^{\mu K} \bar{\psi} \{ \gamma_K , \sigma_{IJ} \} \, .
\end{aligned} \tag{4.19}$$

In the last step above I have used $\{\gamma^I, \sigma_{IJ}\} = 0$ which can be easily verified using the properties of the γ and σ matrices given in Appendix D. I will also discuss the invariance of the Dirac equation in tetrad formulation given by Eq. 3.18 i. e.,

$$\gamma^K e^{\mu}_{K}\, {}^{\psi}\partial_\mu \psi - \frac{i}{4}\omega^{IJ}_{\mu} e^{\mu K}\{\gamma_K, \sigma_{IJ}\}\psi = 0\,. \tag{4.20}$$

Under conformal transformation the above equation goes to

$$\begin{aligned}
&\Omega^{-\frac{5}{2}}\left(\gamma^K e^{\mu}_{K}\, {}^{\psi}\partial_\mu \psi - \frac{i}{4}\omega^{IJ}_{\mu} e^{\mu K}\gamma_K \sigma_{IJ}\psi\right) \\
&\quad - \Omega^{-\frac{5}{2}}\left(\frac{3}{2}\gamma^K e^{\mu}_{K}\psi \partial_\mu \ln\Omega - \frac{i}{2} e^{\nu J}\partial_\nu \ln\Omega\; \gamma^I \sigma_{IJ}\psi\right) = 0 \\
\Rightarrow\quad &\Omega^{-\frac{5}{2}}\left(\gamma^K e^{\mu}_{K}\, {}^{\psi}\partial_\mu \psi - \frac{i}{4}\omega^{IJ}_{\mu} e^{\mu K}\gamma_K \sigma_{IJ}\psi\right) = 0 \\
\Rightarrow\quad &\gamma^K e^{\mu}_{K}\, {}^{\psi}\partial_\mu \psi - \frac{i}{4}\omega^{IJ}_{\mu} e^{\mu K}\gamma_K \sigma_{IJ}\psi = 0\,.
\end{aligned} \tag{4.21}$$

Here I have used $\gamma^I \sigma_{IJ} = 3i\gamma_J$ which again can be verified using the properties of the γ and σ matrices given in Appendix D. The Dirac equation in tetrad formulation is thus invariant under conformal transformation.

In the tetrad formulation discussed above I have not considered torsion anywhere. Let us now go to a broader picture where the connection is not presumed to be torsion-free.

4.2 CONFORMAL TRANSFORMATION IN VIERBEIN-EINSTEIN-PALATINI FORMALISM

I will investigate conformal transformations in the vierbein-Einstein-Palatini formalism in this section. Since the connection is treated an independent variable except when it is on-shell, its conformal properties need to be discussed in two different levels: off-shell and on-shell. By on-shell, I mean that the spin connection and torsion have been replaced by their expressions obtained from the equations of motion. I will discuss this in Sec. 4.3. At the level of the action however, the equations of motion cannot be used. Nevertheless, tetrads and co-tetrads are related to the metric and they transform in the same way as given in Eq. (4.10) and Eq. (4.11).

$$e^I_\mu \to \Omega e^I_\mu, \qquad e^\mu_I \to \Omega^{-1} e^\mu_I\,. \tag{4.22}$$

For the spin connection A_μ^{IJ}, I do not know a priori how it transforms. To find its transformation properties, I will write the transformation of the affine connection $\Gamma^\alpha{}_{\mu\nu}$ in terms of the transformed tetrads and spin connection,

$$\begin{aligned}\Gamma^\alpha{}_{\mu\nu} \to \tilde{\Gamma}^\alpha{}_{\mu\nu} &= \tilde{e}^\alpha_I \partial_\mu \tilde{e}^I_\nu + \tilde{A}^I_{\mu J} \tilde{e}^J_\nu \tilde{e}^\alpha_I \\ &= \delta^\alpha_\nu \partial_\mu (\ln \Omega) + e^\alpha_I \partial_\mu e^I_\nu + \tilde{A}^I_{\mu J} e^J_\nu e^\alpha_I \,.\end{aligned} \tag{4.23}$$

Here the transformed quantities have been denoted with a tilde above them. I can try to determine the transformation of $A^I_{\mu J}$ from this, by positing that the conformally transformed connection $\tilde{\nabla}$ is compatible with the transformed metric $\tilde{g}_{\mu\nu}$. Using Eq. (4.23), we see that

$$\begin{aligned}\tilde{\nabla}_\mu \tilde{g}_{\alpha\beta} &= \partial_\mu \tilde{g}_{\alpha\beta} - \tilde{\Gamma}^\nu{}_{\mu\alpha} \tilde{g}_{\nu\beta} - \tilde{\Gamma}^\nu{}_{\mu\beta} \tilde{g}_{\alpha\nu} \\ &= \partial_\mu (\Omega^2 g_{\alpha\beta}) - 2\Omega^2 \partial_\mu (\ln \Omega) g_{\alpha\beta} - \Omega^2 (\partial_\mu e_{I(\alpha}) e^I_{\beta)} - \Omega^2 \tilde{A}^{IJ}_\mu (e_{J(\alpha} e_{\beta)I}) \\ &= 0\,.\end{aligned} \tag{4.24}$$

In the last equality I have used the orthonormality of the tetrads and the antisymmetry of the spin connection $\tilde{A}^{IJ}_\mu$. Quite clearly, antisymmetry of $\tilde{A}^{IJ}_\mu$ in IJ is sufficient to guarantee metric compatibility. Therefore metric compatibility is not sufficient to determine the transformation of A, and only shows antisymmetry in the internal indices I, J. I am thus at liberty to choose the transformation of the spin connection as long as metricity is satisfied. However, the different possible choices are not guaranteed to reproduce behaviour of usual General Relativity even in the absence of torsion. I will now discuss a couple of such choices and consider matter fields to demonstrate how these choices affect their conformal properties.

4.2.1 *Nieh-Yan theory*

Nieh-Yan theory [23] involves one of the possible choices of conformal transformations of the spin connection. It should be noted that although the spin connection is an independent variable, I can always decompose it in terms of the torsion-free ω, which is completely determined by the tetrads, and the contorsion tensor Λ as defined in Eq. (2.35),

$$A_\mu^{IJ} = \omega_\mu^{IJ} + \Lambda_\mu^{IJ} \,. \tag{4.25}$$

At the level of action λ_μ^{IJ} is independent of the tetrads. In order to find the conformal transformation of A_μ^{IJ}, it should be noted that ω_μ^{IJ} is defined completely in terms of the tetrads regardless of whether it is on-shell or off-shell, and its transformation is given by Eq. (4.12). The independent quantity in the connection is the contorsion component Λ_μ^{IJ} for which I do not know how it transforms off-shell. In other words, I do not have any information about torsion and its transformation. But as discussed above, I can make different choices about the conformal transformation of spin connection as long as metric compatibility is retained. The choice will dictate what physical results I get and also specify the transformation of torsion. The simplest choice in this regard was considered by the authors in [23]. They considered the spin connection to be invariant under conformal transformation,

$$A_\mu^{IJ} \to A_\mu^{IJ} . \tag{4.26}$$

Invariance of the spin connection implies that unlike in General Relativity, here the Riemann tensor and the Ricci tensor remain invariant, while the Ricci scalar transforms homogeneously,

$$R^\rho{}_{\sigma\mu\nu} = F^I_{\mu\nu J} e^\rho_I e^J_\sigma \to F^I_{\mu\nu J} e^\rho_I e^J_\sigma \,, \tag{4.27}$$

$$R_{\mu\nu} = F^I_{\sigma\mu J} e^\sigma_I e^J_\nu \to F^I_{\sigma\mu J} e^\sigma_I e^J_\nu \,, \tag{4.28}$$

$$R = F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \to \Omega^{-2} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \,. \tag{4.29}$$

This is one of the advantages of Nieh-Yan theory, we have a conformally covariant theory of gravity. I will investigate further what the invariance of spin connection implies and how it fixes the transformation of the torsion tensor. It should be noted that in order for the spin connection A_μ^{IJ} of Eq. (4.25) to remain invariant, the transformation of contorsion part Λ must cancel that of the torsion-free part ω given by Eq. (4.12), i. e.,

$$\Lambda_\mu^{IJ} \to \Lambda_\mu^{IJ} - (e^I_\mu e^\nu_J - e_{\mu J} e^{\nu I}) \partial_\nu \ln \Omega \,. \tag{4.30}$$

The transformation of torsion tensor, which is given in terms of Λ in Eq. (2.36), is thus

$$^{NY}C^{\alpha}{}_{\mu\nu} \rightarrow ^{NY} C^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{\nu}\partial_{\mu}\ln\Omega - \delta^{\alpha}_{\mu}\partial_{\nu}\ln\Omega\,. \tag{4.31}$$

Clearly the Nieh-Yan torsion tensor $^{NY}C^{\alpha}{}_{\mu\nu}$ transforms inhomogeneously, or in other words, torsion acts as a gauge field in the conformal transformation group, which is a fundamental result of Nieh-Yan theory. I will now consider the fermionic field and the scalar field in this theory.

4.2.1.1 *Fermionic field in Nieh-Yan theory*

Because I am working with actions, I will consider the fermionic Lagrangian rather than the Dirac equation. The Lagrangian of the fermionic field given in Eq. (3.21),

$$\mathcal{L}_F = \frac{i}{2}\left(\bar{\psi}\gamma^K e^{\mu}_K \partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^K e^{\mu}_K\psi - \frac{i}{4}A^{IJ}_{\mu}e^{\mu K}\bar{\psi}\{\gamma_K,\sigma_{IJ}\}\psi\right)\,, \tag{4.32}$$

transforms homogeneously with conformal weight of -4 if the fermion ψ is taken to transform in the same way as in Eq. (4.18), provided A^{IJ}_{μ} is invariant as in Nieh-Yan theory. The Dirac equation can thus be expected to remain conformally covariant too. But there are certain problems with the equation as we will see now. Let us recall the Dirac equation in the vierbein-Einstein-Palatini formalism, which is given in Eq. (3.29),

$$\gamma^K e^{\mu}_K\, {}^{\psi}D_\mu\psi = 0\,. \tag{4.33}$$

If the spin connection remains invariant under conformal transformations, this equation transforms to

$$\Omega^{-\frac{5}{2}}\left(\gamma^K e^{\mu}_K\, {}^{\psi}D_\mu\psi - \frac{1}{2}\gamma^K e^{\mu}_K\psi\partial_\mu\ln\Omega\right) = 0\,. \tag{4.34}$$

Clearly, the equation is not invariant under these transformations. The source of this problem lies in the fact that the spinor equation as written here was obtained after using the total antisymmetry of the on-shell torsion. Let us see how this equation transforms if I do not use any on-shell property of torsion that is derived from the equations of motion. In this case I need to consider Eq. (3.23), i. e.,

$$\gamma^K e^{\mu}_K\partial_\mu\psi + \frac{1}{2}C^{\alpha}{}_{\mu\alpha}e^{\mu}_K\gamma^K\psi - \frac{i}{4}A^{IJ}_{\mu}e^{\mu K}\gamma_K\sigma_{IJ}\psi = 0\,. \tag{4.35}$$

Replacing ${C^\alpha}_{\mu\alpha}$ with Nieh-Yan torsion ${}^{NY}{C^\alpha}_{\mu\alpha}$, the above equation becomes

$$\gamma^K e^\mu_K \partial_\mu \psi + \frac{1}{2}\,{}^{NY}{C^\alpha}_{\mu\alpha} e^\mu_K \gamma^K \psi - \frac{i}{4} A^{IJ}_\mu e^{\mu K} \gamma_K \sigma_{IJ} \psi = 0\,. \tag{4.36}$$

The transformation of ${}^{NY}{C^\alpha}_{\mu\alpha}$ can be obtained from Eq. (4.31) by tracing over first and the last index of the torsion tensor,

$${}^{NY}{C^\alpha}_{\mu\alpha} \rightarrow {}^{NY}{C^\alpha}_{\mu\alpha} + 3\partial_\mu \ln \Omega\,. \tag{4.37}$$

Taking the above into account, we can see that the Dirac equation remains conformally covariant. It is thus clear that if on-shell properties of torsion are used, the Dirac equation does not remain invariant under the assumptions of Nieh-Yan theory. We will see in Sec. 4.3 that on-shell torsion does not transform in the way given in Eq.4.31 above.

4.2.1.2 *Conformal scalar in Nieh-Yan theory*

I will start by writing the Lagrangian of the conformal scalar field in terms of vierbein-Einstein-Palatini variables,

$$\mathcal{L}_\phi = -\frac{1}{2} e^\mu_I e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2\,. \tag{4.38}$$

In Nieh-Yan theory the torsion tensor that transforms inhomogeneouly as shown in Eq. (4.31), bu $F^{IJ}_{\mu\nu}$ transforms homogeneously as in Eq. (4.29). This above Lagrangian is not covariant as a result, with $\phi \rightarrow \Omega^{-1}\phi$.

The Lagrangian can be rewritten with the torsion-free part of $F^{IJ}_{\mu\nu}$ in order to make it conformally covariant,

$$\begin{aligned}\mathcal{L}_\phi = -\frac{1}{2} e^\mu_I e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2 - \phi^2 \Bigg(\frac{1}{6} \widehat{\nabla}_\mu\, {}^{NY}{C^\alpha}_\alpha{}^\mu + \frac{1}{12}\, {}^{NY}{C^\mu}_{\mu\sigma}\, {}^{NY}{C^\nu}_\nu{}^\sigma \\ - \frac{1}{48}\, {}^{NY}C^{\mu\nu\sigma}\, {}^{NY}C_{\mu\nu\sigma} - \frac{1}{24}\, {}^{NY}C^{\mu\nu\sigma}\, {}^{NY}C_{\nu\mu\sigma} \Bigg)\,.\end{aligned} \tag{4.39}$$

We can see that the coefficients of ϕ^2 in the above add up to produce the Ricci scalar $\widehat{R}$ corresponding to the torsion-free connection. The Lagrangian can thus be written as

$$\mathcal{L}_\phi = -\frac{1}{2} e^\mu_I e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} \widehat{F}^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2\,. \tag{4.40}$$

This is the same Lagrangian of conformal scalar of Eq. (4.15) in tetrad formulation and transforms homogeneously with conformal weight of -4. Evidently the corresponding equation remains conformally invariant.

It should be noted that I have written the Lagrangian given by Eq. (4.39) with the full torsion-free Ricci scalar so that in the torsion-free limit, the Lagrangian remains conformally covariant. But if I am not concerned about the torsion-free limit, the minimum modification of the Lagrangian is

$$\mathcal{L}_\phi = -\frac{1}{2}e^\mu_I e^{\nu I}\partial_\mu\phi\partial_\nu\phi - \phi^2\left(\frac{1}{6}\widehat{\nabla}_\mu\, {}^{NY}C^{\alpha}{}_{\alpha}{}^{\mu} + \frac{1}{12}\, {}^{NY}C^{\mu}{}_{\mu\sigma}\, {}^{NY}C^{\nu}{}_{\nu}{}^{\sigma} \right.$$
$$\left. -\frac{1}{48}\, {}^{NY}C^{\mu\nu\sigma}\, {}^{NY}C_{\mu\nu\sigma} - \frac{1}{24}\, {}^{NY}C^{\mu\nu\sigma}\, {}^{NY}C_{\nu\mu\sigma}\right) . \tag{4.41}$$

This Lagrangian transforms homogeneously with conformal weight of -4 under conformal transformation. I have removed the $\frac{1}{12}F^{IJ}_{\mu\nu}e^\mu_I e^\nu_J\phi^2$ from the Lagrangian given in Eq. (4.39) because this term transforms homogeneously and does not affect the conformal covariance of the Lagrangian in Nieh-Yan theory. The corresponding scalar field equation

$$\nabla_\mu\nabla^\mu\phi - \phi\left(\frac{1}{3}\widehat{\nabla}_\mu\, {}^{NY}C^{\alpha}{}_{\alpha}{}^{\mu} + \frac{1}{6}\, {}^{NY}C^{\mu}{}_{\mu\sigma}\, {}^{NY}C^{\nu}{}_{\nu}{}^{\sigma} - \frac{1}{24}\, {}^{NY}C^{\mu\nu\sigma}\, {}^{NY}C_{\mu\nu\sigma}\right.$$
$$\left. -\frac{1}{12}\, {}^{NY}C^{\mu\nu\sigma}\, {}^{NY}C_{\nu\mu\sigma}\right) = 0 . \tag{4.42}$$

is also conformally invariant. Clearly in the torsion-free limit the torsion terms in the above Lagrangian and equation vanish. Consequently the conformal covariance (or invariance) gets lost when torsion vanishes. This reinstates my previous statement that torsion-free limit, not all the results and equations of Nieh-Yan theory reduce to the corresponding results and equations in General Relativity.

4.2.2 *Conformally invariant torsion*

Although Nieh-Yan theory gives a conformally covariant theory of gravity, not all its results can be identified with those in Einstein's General Relativity in the absence of torsion. For example the conformal transformations of different quantities do not reduce to those of General Relativity when torsion vanishes as we have seen in case of conformal scalar in the previous subsection. I am interested in a formalism which resembles Einstein gravity and in which different entities transform in the same way as in usual General Relativity when torsion is taken to vanish. In such a theory, I must assume that torsion transforms homogeneously under conformal transformation, unlike in [23] where torsion acts a gauge potential in conformal group. In other words, the spin connection A_μ^{IJ} should transform in the same way as ω_μ^{IJ} does while Λ_μ^{IJ} should remain invariant i.e.,

$$A_\mu^{IJ}(\equiv \omega_\mu^{IJ} + \Lambda_\mu^{IJ}) \to A_\mu^{IJ} + (e_\mu^I e^{J\alpha} - e_\mu^J e^{I\alpha})\partial_\alpha \ln \Omega\,, \tag{4.43}$$

and

$$\Lambda_\mu^{IJ} \to \Lambda_\mu^{IJ}\,. \tag{4.44}$$

In this case torsion in non-zero but conformal transformation is given by that of tetrads only. Above transformations immediately imply that torsion tensor given by the simplified form in Eq. (2.36), remains invariant i. e.,

$${}^{Inv}C^\alpha{}_{\mu\nu} \to {}^{Inv}C^\alpha{}_{\mu\nu}\,. \tag{4.45}$$

It should be noted that if any index of the torsion tensor is raised or lowered, there will be a conformal weight factor due to the metric involved in the raising or lowering. The Ricci scalar contains extra terms involving torsion when transformed,

$$\begin{aligned} F_{\mu\nu}^{IJ} e_I^\mu e_J^\nu \to \Omega^{-2} \Big[& F_{\mu\nu}^{IJ} e_I^\mu e_J^\nu - 6\widehat{\nabla}_\mu \widehat{\nabla}^\mu \ln \Omega - 6 \left(\widehat{\nabla}_\mu \ln \Omega\right) \left(\widehat{\nabla}^\mu \ln \Omega\right) \\ & -4\, {}^{Inv}C^\alpha{}_\alpha{}^\mu \partial_\mu \ln \Omega \Big]\,. \end{aligned} \tag{4.46}$$

It is interesting to note that although torsion itself does not transform, it appears as a part of the transformation. I will now look at the fermionic and scalar field in the presence of invariant torsion.

4.2.2.1 *Conformal properties of fermionic field with invariant torsion*

Let us recall the Lagrangian of the fermionic field,

$$\mathcal{L}_F = \frac{i}{2}\left(\bar{\psi}\gamma^K e_K^\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^K e_K^\mu \psi - \frac{i}{4} A_\mu^{IJ} e^{\mu K} \bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\right)\,. \tag{4.47}$$

In order to see how this Lagrangian transforms, it should be noted that with the transformation of the spin connection given by Eq. (4.43), the last term in the above Lagrangian remains unchanged as we have seen in section 4.1.2. The inhomogeneous transformations of the other two terms cancel each other and as a result, the Lagrangian transforms homogeneously with conformal weight of -4 as before. Let us now see what the invariant torsion implies for the conformal invariance of the Dirac equation.

With the spin connection transforming inhomogeneously given by Eq. (4.43), we can see that the term containing the spin connection in the Dirac equation of Eq. (3.28) transforms as

$$A_\mu^{IJ} e^{\mu K} \gamma_K \sigma_{IJ} \psi \quad \to \quad \Omega^{-\frac{5}{2}} (A_\mu^{IJ} e^{\mu K} \gamma_K \sigma_{IJ} \psi + 2 i e^{\mu K} \gamma_K \psi \partial_\mu \ln \Omega) \, . \tag{4.48}$$

This implies that the Dirac equation remains invariant with $\psi \to \Omega^{-\frac{3}{2}} \psi$, although the equation uses the skew symmetry of on-shell torsion. This is a difference with what was observed in Nieh-Yan theory, where I showed that using the on-shell expression of torsion breaks the conformal invariance of the Dirac equation.

4.2.2.2 *Conformal scalar with invariant torsion*

Let us now look at the conformal scalar with invariant torsion. As in Nieh-Yan theory I will start by writing the Lagrangian of the conformal scalar in terms of the vierbein-Einstein-Palatini variables,

$$\mathcal{L}_\phi = -\frac{1}{2} e_I^\mu e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} F_{\mu\nu}^{IJ} e_I^\mu e_J^\nu \phi^2 \, . \tag{4.49}$$

Using the transformation of the Ricci scalar given by Eq. (4.46), we see that the above Lagrangian transforms as

$$\mathcal{L}_\phi \quad \to \quad \Omega^{-4} \mathcal{L}_\phi + \frac{1}{3} \Omega^{-4} \phi^2 \, {}^{Inv}C^\alpha{}_\alpha{}^\mu \partial_\mu \ln \Omega \, . \tag{4.50}$$

The minimal modification that makes the Lagrangian conformally covariant is the addition of a torsion term,

$$\mathcal{L}_\phi = -\frac{1}{2} e_I^\mu e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} F_{\mu\nu}^{IJ} e_I^\mu e_J^\nu \phi^2 - \frac{1}{6} \phi^2 \widehat{\nabla}_\mu \, {}^{Inv}C^\alpha{}_\alpha{}^\mu \, . \tag{4.51}$$

This Lagrangian transforms homogeneously because the added term has the following transformation.

$$\frac{1}{6}\phi^2\widehat{\nabla}_\mu \ {}^{Inv}C^{\alpha}{}_{\alpha}{}^{\mu} \quad \rightarrow \quad \Omega^{-4}\left(\frac{1}{6}\phi^2\widehat{\nabla}_\mu \ {}^{Inv}C^{\alpha}{}_{\alpha}{}^{\mu} + \frac{1}{3}\phi^2 \ {}^{Inv}C^{\alpha}{}_{\alpha}{}^{\mu}\partial_\mu \ln\Omega\right). \tag{4.52}$$

The above might seem to suggest that torsion transforms inhomogeneously although I am considering invariant torsion. It should be noted that $C^{\alpha}{}_{\mu\nu}$ is conformally invariant. Raising or lowering of indices involves the metric and thus results in a conformal weight e. g.,

$$C^{\alpha}{}_{\mu}{}^{\beta} \quad \equiv \quad g^{\beta\nu}C^{\alpha}{}_{\mu\nu} \quad \rightarrow \quad \Omega^{-2}C^{\alpha}{}_{\mu}{}^{\beta}. \tag{4.53}$$

Taking a derivative with respect to $\widehat{\nabla}$ thus results in an inhomogeneous transformation. Also the transformation of $\widehat{\Gamma}^{\alpha}{}_{\mu\nu}$ adds to the inhomogeneous part. The scalar equation corresponding to the Lagrangian given in Eq. (4.51)

$$\widehat{\nabla}_\mu\widehat{\nabla}^\mu\phi - \frac{1}{6}F^{IJ}_{\mu\nu}e^\mu_I e^\nu_J\phi - \frac{1}{3}\phi\widehat{\nabla}_\mu \ {}^{Inv}C^{\alpha}{}_{\alpha}{}^{\mu} = 0\,. \tag{4.54}$$

is invariant under conformal transformation. In the torsion-free limit, the above equation and Lagrangian given in Eq. (4.51) reduce to their counterparts in the tetrad formulation of General Relativity. Thus unlike in case of Nieh-Yan theory, here the the torsion-free limit reduces to General Relativity.

It should be further noted that I can also write the torsion-free part of the Ricci scalar with quadratic torsion terms, as I did in the case of Nieh-Yan theory, without affecting the covariance of the Lagrangian or invariance the equation. This is because the quadratic torsion terms transform homogeneously with conformal weight of -2.

I can thus say that the general Lagrangian for the conformally invariant scalar field is

$$\begin{aligned}\mathcal{L}_\phi = -\frac{1}{2}e^\mu_I e^{\nu I}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}F^{IJ}_{\mu\nu}e^\mu_I e^\nu_J\phi^2 \\ -\phi^2\left(\frac{1}{6}\widehat{\nabla}_\mu \ {}^{Inv}C^{\alpha}{}_{\alpha}{}^{\mu} + \frac{1}{12} \ {}^{Inv}C^{\mu}{}_{\mu\sigma} \ {}^{Inv}C^{\nu}{}_{\nu}{}^{\sigma}\right. \\ \left. -\frac{1}{48} \ {}^{Inv}C^{\mu\nu\sigma} \ {}^{Inv}C_{\mu\nu\sigma} - \frac{1}{24} \ {}^{Inv}C^{\mu\nu\sigma} \ {}^{Inv}C_{\nu\mu\sigma}\right).\end{aligned} \tag{4.55}$$

Similar to what was obtained in Nieh-Yan theory, the coefficients of ϕ^2 can be identified as $-\frac{1}{12}\widehat{R}$. This implies that although I am dealing with non-zero off-shell torsion, for the conformally invariant scalar field the Lagrangian can always be written with torsion-free Ricci scalar.

4.3 DYNAMICALLY GENERATED TORSION AND CONFORMAL TRANSFORMATION

In this section I will deal with dynamically generated (on-shell) torsion and see its effects on conformal properties of matter fields. Torsion is sourced from other dynamical fields and it comes from the equations of the spin connection. So nothing is assumed a priori about the transformation of torsion. We have seen in chapter 3 that spinor fields can produce torsion. I will demonstrate how non-minimal scalar fields can also produce torsion on-shell. But there are problems with the conformal weight of the torsion terms if they are dynamically generated. We will see that on-shell torsion transforms homogeneously but unlike invariant torsion, its transforms with an overall weight. I will demonstrate these problems and possible solutions with specific fields.

4.3.1 *Dynamically generated torsion and conformal scalar*

The total action including the conformal scalar, in terms of vierbein-Einstein-Palatini variables is given by

$$S[e,A,\phi] = S_{VEP}[e,A] + \int |e| d^4x \left[-\frac{1}{2} e^\mu_I e^{\nu I} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2 \right] . \tag{4.56}$$

Extremising the action with respect to the independent variables produces three sets of equations,

$$\delta e^\nu_J \; : \; F^{IJ}_{\alpha\mu} e^\alpha_I e_{\nu J} - \frac{1}{2} g_{\mu\nu} F^{IJ}_{\alpha\beta} e^\alpha_I e^\beta_J = \kappa \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right.$$
$$\left. + \frac{1}{6} \left(F^{IJ}_{\alpha\mu} e^\alpha_I e_{\nu J} - \frac{1}{2} g_{\mu\nu} F^{IJ}_{\alpha\beta} e^\alpha_I e^\beta_J \right) \right) , \tag{4.57a}$$

$$\delta A^{IJ}_\nu \; : \; A^{IJ}_\mu = \omega^{IJ}_\mu [e] + \frac{1}{2} \left(e^I_\mu e^{J\alpha} - e^J_\mu e^{I\alpha} \right) \partial_\mu \ln \left(1 - \frac{\kappa \phi^2}{6} \right) , \tag{4.57b}$$

$$\delta \phi \; : \; \widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi = 0 \, . \tag{4.57c}$$

Here I have contracted the equations with tetrads in order to convert them to familiar forms. It should be noted that in Eq. (4.57c), the first term has been written in terms of the torsion-free derivative since

$$\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \nabla_\mu \nabla^\mu \phi = S^{\alpha\mu}{}_\alpha \partial_\mu \phi = 0\,, \tag{4.58}$$

which follows from the definition of contorsion tensor given in Eq. (1.17). The right hand side of Eq. (4.57a) does not contain the full energy-momentum tensor as can be seen on comparison with Eq. (4.8). As a result it will not be a conserved tensor. This is because in this case the Einstein tensor $G_{\mu\nu}$, which appears on the left hand side of the equation, is torsionful. I can separate the torsion-free and torsionful parts by considering the on-shell expression of the spin connection (4.57b). Before getting into this, let us first see how Eq. (4.57b) can lead to non-zero torsion. The spin connection up to $\mathcal{O}(\kappa)$ can be written as

$$A_\mu^{IJ} \approx \omega_\mu^{IJ}[e] - \frac{\kappa}{12}\left(e_\mu^I e^{J\alpha} - e_\mu^J e^{I\alpha}\right)\partial_\mu \phi^2\,. \tag{4.59}$$

Comparing with Eq. (2.35), the second term in this equation can be identified as contorsion Λ,

$$\Lambda_\mu^{IJ} = -\frac{\kappa}{12}\left(e_\mu^I e^{J\alpha} - e_\mu^J e^{I\alpha}\right)\partial_\mu \phi^2\,. \tag{4.60}$$

This gives the following expression for the on-shell torsion tensor

$$^{OS}C^\alpha{}_{\mu\nu} = \frac{\kappa}{12}\left(\delta_\mu^\alpha \partial_\nu \phi^2 - \delta_\nu^\alpha \partial_\mu \phi^2\right)\,. \tag{4.61}$$

It should be noted here that the expressions of on-shell torsion found Eq. (4.61) have been found in other contexts like conformal Standard Model [56]. I have obtained the expression for on-shell torsion from the equation of the spin connection whereas in [56], the conformal transformation of the torsion tensor was postulated in a form which seems to come from the expression given in Eq. (4.61). Inserting Eq. (4.59) in Eq. (4.57a) gives

$$\begin{aligned}\widehat{F}^{IJ}_{\alpha\mu} e_I^\alpha e_{\nu J} - \frac{1}{2} g_{\mu\nu} \widehat{F}^{IJ}_{\alpha\beta} e_I^\alpha e_J^\beta = \kappa \Bigg(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}(\widehat{F}^{IJ}_{\alpha\mu} e_I^\alpha e_{\nu J} \\ -\frac{1}{2} g_{\mu\nu}\widehat{F}^{IJ}_{\alpha\beta} e_I^\alpha e_J^\beta)\phi^2 + \frac{1}{6}\left[g_{\mu\nu}\widehat{\nabla}_\sigma\widehat{\nabla}^\sigma\phi^2 - \widehat{\nabla}_\mu\widehat{\nabla}_\nu\phi^2\right]\Bigg) + \mathcal{O}(\kappa^2)\,.\end{aligned} \tag{4.62}$$

I have thus obtained the correct energy-momentum tensor, with $\mathcal{O}(\kappa^2)$ contributions which can be neglected. There is however another problem with Eq. (4.57c). I can insert Eq. (4.59) in Eq. (4.57c) to write it up to $\mathcal{O}(\kappa)$ as

$$\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6}\widehat{F}^{IJ}_{\alpha\beta} e^\alpha_I e^\beta_J \phi + \frac{\kappa}{24}\phi \widehat{\nabla}_\alpha \widehat{\nabla}^\alpha \phi^2 = 0\,. \tag{4.63}$$

Upon a conformal transformation this equation becomes

$$\Omega^{-3}\left(\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6}\widehat{R}\phi\right) + \frac{\kappa}{24}\Omega^{-5}\phi \widehat{\nabla}_\alpha \widehat{\nabla}^\alpha \phi^2 - \frac{\kappa}{6}\Omega^{-5}\phi^2 g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ - \frac{\kappa}{12}\Omega^{-5}\phi^3 \widehat{\nabla}_\alpha \widehat{\nabla}^\alpha \ln\Omega = 0\,. \tag{4.64}$$

Clearly the equation is not conformally invariant. The last term in Eq. (4.63) not only has different conformal weight, it also transforms inhomogeneously. The source of this discrepancy lies in the expression of spin connection, more precisely the contorsion Λ, which transforms as

$$\Lambda^{IJ}_\mu \to \Omega^{-2}\Lambda^{IJ}_\mu + \frac{\kappa}{6}\Omega^{-2}\left(e^I_\mu e^{J\alpha} - e^J_\mu e^{I\alpha}\right)\phi^2 \partial_\alpha \ln\Omega\,. \tag{4.65}$$

The conformal weight of Λ^{IJ}_μ is different from that of the torsion-free part ω^{IJ}_μ of Eq. (4.12) and also it transforms inhomogeneously. I can resolve this problem by modifying the action such that it does not produce torsion dynamically. The connection used in the vierbein-Einstein-Palatini formalism is not torsion-free a priori. However, the $R\phi^2$ term in the metric formalism, is constructed from a torsion-free connection. Therefore, in the vierbein-Einstein-Palatini formalism if I construct the non-minimal $R\phi^2$ term only from the torsionless part of the connection I can eliminate on-shell torsion. The total action now reads

$$S[e, A, \phi] = \frac{1}{2\kappa}\int |e| d^4x F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J + \int |e| d^4x \left(-\frac{1}{2} e^\mu_I e^{\nu I}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}\widehat{F}[\omega]^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2\right)\,. \tag{4.66}$$

The equations of motion as obtained from this action are

$$\delta e^{\nu}_{J} : R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{6} (\widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R}) \phi^2 \right.$$
$$\left. + \frac{1}{6} \left[g_{\mu\nu} \widehat{\nabla}_\sigma \widehat{\nabla}^\sigma \phi^2 - \widehat{\nabla}_\mu \widehat{\nabla}_\nu \phi^2 \right] \right), \tag{4.67a}$$

$$\delta A^{IJ}_{\nu} : A^{IJ}_{\mu} = \omega^{IJ}_{\mu}[e], \tag{4.67b}$$

$$\delta \phi : \widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6} \widehat{R} \phi = 0\,. \tag{4.67c}$$

The scalar field equation (4.67c), thus gets back the torsion-free, conformally invariant form given by Eq. (4.5). The action in Eq. (4.66) leads to vanishing on-shell torsion and the equations can be identified with those in the usual metric formalism. Also upon using the on-shell expression of the spin connection given by Eq. (4.67b), the left hand side of Einstein's equation (4.67a) becomes torsion-free and the equation can be identified with that in the tetrad formalism given in Eq. (4.16). I have also discussed an alternative but equivalent method of obtaining conformally invariant scalar field equation in Appendix B.

4.3.2 *Dynamically generated torsion and fermion*

I will now consider the conformal transformation of fermionic field equation when the on-shell torsion arises from the field itself. The expression for the on-shell torsion with fermionic field was found in Eq. (3.27), and using this expression, I obtained a nonlinear Dirac equation

$$\gamma^K e^\mu_K \partial_\mu \psi - \frac{i}{4} \omega^{IJ}_\mu e^{\mu K} \gamma_K \sigma_{IJ} \psi - \frac{i\kappa}{64} \bar{\psi} \{\gamma_K, \sigma_{IJ}\} \psi \{\gamma^K, \sigma^{IJ}\} \psi = 0\,. \tag{4.68}$$

Comparing with the Dirac equation in the tetrad formulation, I recognize that the first two terms are covariant under conformal transformations. The cubic term however transforms with a different weight, as can be seen from the transformed equation,

$$\Omega^{-\frac{5}{2}} \left(\gamma^K e^\mu_K \partial_\mu \psi - \frac{i}{4} \omega^{IJ}_\mu e^{\mu K} \gamma_K \sigma_{IJ} \psi \right) - \frac{i\kappa}{64} \Omega^{-\frac{9}{2}} \bar{\psi} \{\gamma_K, \sigma_{IJ}\} \psi \{\gamma^K, \sigma^{IJ}\} \psi = 0\,. \tag{4.69}$$

The nonlinear Dirac equation is thus not invariant under conformal transformation. The source of this difference in conformal weight is as follows.

The cubic term which breaks the invariance of the equation appeared due to the dynamically generated contorsion Λ. I will consider the transformation of the spin connection of Eq. (3.22b) to see this.

$$A_\mu^{IJ} \quad \rightarrow \quad \omega_\mu^{IJ}[e] + \left(e_\mu^I e^{J\nu} - e_\mu^J e^{I\nu}\right) \partial_\nu \ln \Omega + \Omega^{-2} \Lambda_\mu^{IJ} \,. \tag{4.70}$$

The two components of the spin connection ω and $\Lambda(= \frac{\kappa}{8} \bar{\psi}\{\gamma_K, \sigma^{IJ}\}\psi e_\mu^K)$ transform with different conformal weights. This is because they come from two different sectors of the theory. While ω comes from the gravity sector and is fully determined by the geometry, Λ comes from the matter sector, in this case fermions. But these two sectors have different conformal weights. The fermion Lagrangian transforms homogeneously with weight -4 but R in the gravity part has weight of -2. This difference shows up in the transformation of Λ. The transformation of the spin connection also implies that torsion as given by Eq. (3.27) transforms homogeneously with conformal weight -2,

$${}^{OS}C^\nu{}_{\rho\lambda} \rightarrow \Omega^{-2}\, {}^{OS}C^\nu{}_{\rho\lambda} \,. \tag{4.71}$$

It is thus clear that on-shell torsion tensor (with index positions as above) transforms homogeneously unlike in Nieh-Yan theory where it has an inhomogeneous transformation. Also the above transformation is different from that of the invariant torsion of Eq. (4.45) because it transforms with a factor of Ω^{-2}.

I can thus conclude that similar to what has been observed in the case of scalar field, dynamically generated torsion also breaks the conformal invariance of Dirac equation. I can try to make the Dirac equation conformally invariant but this requires the fundamental theories to be modified as I will discuss below.

There are different ways to proceed in order to reinstate conformal invariance. First, if I have an action for gravity that scales in the same manner as the matter action, I can eliminate the above mentioned weight difference of ω and Λ altogether. A scale invariant theory of gravity with spinors, as discussed in [57] where the authors considered Brans-Dicke (BD) theory of gravity, might be able to provide a way out. If it is possible to construct such a theory of gravity without affecting the on-shell quantities obtained, I will get back the conformally covariant Dirac equation. Another way was discussed in [58] where an invariant theory of Dirac field was considered in conformal gravity such that the nonlinearities in the Dirac equation do not appear.

A third way of making the Dirac equation conformally invariant is to modify the fermionic Lagrangian. If I am interested in the conformal invariance of the Dirac equation alone, the spinor Lagrangian can be modified by the addition of a quartic term. The term should be chosen in such a way that it cancels the cubic term thereby making the equation linear. It should also be kept in mind that this term should not affect the expression for on-shell torsion i. e., the spin connection should not appear in it. I will modify the spinor Lagrangian as

$$\begin{aligned}\mathcal{L}_{F-mod} = \frac{i}{2}\Bigg(\bar{\psi}\gamma^K e^\mu_K \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^K e^\mu_K \psi - \frac{i}{4} A^{IJ}_\mu e^{\mu K} \bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi \\ + \frac{i\kappa}{64}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\bar{\psi}\{\gamma^K, \sigma^{IJ}\}\psi\Bigg)\,.\end{aligned} \tag{4.72}$$

The equation obtained by extremising the corresponding action with respect to $\bar{\psi}$ is

$$\gamma^K e^\mu_K \partial_\mu \psi - \frac{i}{4} A^{IJ}_\mu e^{\mu K}\gamma_K \sigma_{IJ}\psi + \frac{i\kappa}{64}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\{\gamma^K, \sigma^{IJ}\}\psi = 0\,. \tag{4.73}$$

Also, extremisation with the spin connection produces exactly the same expressions for the contorsion Λ and torsion C as found in Eq. (3.24) and Eq. (3.27) earlier. So when putting the on-shell expression of the spin connection in the above equation gives nothing but the linear Dirac equation which was obtained in tetrad formulation i. e.,

$$\gamma^K e^\mu_K \partial_\mu \psi - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\gamma_K \sigma_{IJ}\psi = 0\,. \tag{4.74}$$

I have thus eliminated the cubic term and obtained the conformally invariant Dirac equation. The added term however, also cancels the $\mathcal{O}(\kappa^2)$ on the right hand side of Einstein's equations (3.31). In other words, with the addition of the term I am effectively dealing with a torsion-free theory.

Also, unlike in Nieh-Yan theory (and invariant torsion), the Lagrangian above is not conformally invariant although the equation it produces is invariant under conformal transformation.

4.4 GENERAL OFF-SHELL TRANSFORMATION

I have discussed two possibilities of the transformation of the spin connection A_μ^{IJ} in the level of action. The two cases can be parametrised with a single parameter. In order to do that, I will write torsion-free part of the connection as

$$\omega_\mu^{IJ} = A_\mu^{IJ} - \Lambda_\mu^{IJ} \,. \tag{4.75}$$

The transformation of ω is dictated from that of the tetrads and this implies that the right hand side of the equation has a definite conformal transformation. If I consider the case where A_μ^{IJ} remains invariant, Λ_μ^{IJ} must transform like ω_μ^{IJ} with a negative sign (Nieh-Yan). On the other hand, if it is Λ_μ^{IJ} that remains invariant, A_μ^{IJ} must transform like ω_μ^{IJ} (invariant torsion). The transformations

$$A_\mu^{IJ} \to A_\mu^{IJ} + \xi \left(e_\mu^I e^{J\nu} - e_\mu^J e^{I\nu} \right) \partial_\nu \ln \Omega \,, \tag{4.76}$$

$$\Lambda_\mu^{IJ} \to \Lambda_\mu^{IJ} - (1-\xi) \left(e_\mu^I e^{J\nu} - e_\mu^J e^{I\nu} \right) \partial_\nu \ln \Omega \,, \tag{4.77}$$

interpolate between the two cases with $\xi = 0$ and $\xi = 1$, with Nieh-Yan theory corresponding to $\xi = 0$ and invariant torsion corresponding to $\xi = 1$. Other values of ξ also correspond to conformal transformations, with the Dirac Eq. (3.23) remaining invariant for any value of ξ between 0 and 1. This parametrization may be compared with that postulated for the transformation of torsion in conformal gravity [59], where it is an undetermined constant. For the scalar field however, the non-minimal coupling term $-\frac{1}{12}R\phi^2$ must be written using the torsion-free Ricci scalar in order for the field equation to be conformally invariant, as was observed for both Nieh-Yan theory and invariant torsion.

I will conclude this chapter with a few remarks. We have seen that dynamically generated torsion does not transform inhomogeneously at all. There is of course an issue with torsion if considered to be given by the equations of motion alone: the contorsion part has a relative weight over the torsion-free part.

For minimally coupled fermionic fields, the on-shell torsion resulting from the fermion coupling makes the Dirac equation nonlinear. This equation is not conformally invariant. We have seen that there are different ways of restoring conformal invariance of the Dirac equation, one of which involves the addition of a quartic term in the Lagrangian. This term helps to recover the conformally invariant linear Dirac equation by setting the torsion to vanish on shell, but the cost is the additional quartic term. It should be noted that it is also possible to consider the torsion, or alternatively fermion, as a source of explicit breaking of conformal symmetry, as was done in [60].

5

NEUTRINO MIXING

We have seen in chapter 3 that torsion induces an effective self-interaction among fermions thereby making the Dirac equation nonlinear, analogous to the Nambu-Jona-Lasinio model [61, 62]. This feature has been exploited in particle physics, for example in [63] it has been suggested that torsion induces interactions among leptons identical to the weak leptonic interactions in Weinberg's standard model [64]. The quartic interaction induced by minimal coupling of fermions with torsion has been shown to help replace the big bang singularity with a cusp-like bounce [65]. In cosmological models [66–68] the possibility of a self accelerating universe has been discussed with the help of torsion. Also it has been suggested that the four-fermion interaction originating from spin-torsion coupling can be seen as dark energy [69]. The possibility of inflationary phase in the early universe has been discussed using only spin and torsion without the need of any extra fields [70]. In this chapter[1] I will use the four fermion interaction to explain the source of neutrino mass and oscillation.

Neutrinos were originally thought to be massless charge neutral particles. However in order to explain the flavour oscillations of neutrinos, it is necessary to assume that neutrinos have mass eigenstates different from their flavour eigenstates [71]. But the origin of neutrino mass is a mystery [72, 73]. The Standard Model of particle physics is successful because it encompasses all the known elementary particles and the interactions between them, explains the masses of elementary particles. Particles get mass in terms of electroweak symmetry breaking and the vacuum expectation value of the Higgs field. The Higgs field in the electroweak theory is a complex doublet whose potential reaches a local minimum for a continuous range of configurations of the field, corresponding to a non-vanishing vacuum expectation value (vev) of the neutral scalar Higgs field. This is the phenomenon of spontaneous symmetry breaking, which means

1 The work reported here is based on the paper "Geometrical contribution to neutrino mass matrix", Subhasish Chakrabarty and Amitabha Lahiri, Eur. Phys. J. C 79, 8, 697 (2019).

that the vacuum is not invariant under the symmetry transformations of the classical Lagrangian and the quantum Higgs field consists of fluctuations around this vev.

In the Standard Model, it is the $SU(2) \times U(1)$ electroweak symmetry which is spontaneously broken in the vacuum. For the fermions, this symmetry is a chiral one. Let us focus on the leptons, but what I say can be generalized to quarks quite easily. Left-handed components of leptons pair up into doublets of weak isospin $\Psi_{eL} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ while a right-handed component has never been observed for the neutrino and thus the right-handed electron e_R is by necessity a singlet. The Higgs doublet field $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ couples left-handed doublets to the right-handed singlet via the Yukawa-type interaction

$$-h_e \left(\bar{\Psi}_{eL} \Phi e_R + \bar{e}_R \Phi^\dagger \Psi_{eL} \right) . \tag{5.1}$$

For quantisation, ϕ^0 is expanded around its vev v as $\phi^0 = \frac{1}{\sqrt{2}}(v + H + i\zeta)$ with H, ζ being quantum fields. Then the Yukawa terms can be written as

$$-h_e \left[\frac{v}{\sqrt{2}} (\bar{e}_L e_R + \bar{e}_R e_L) + \bar{\nu}_{eL} e_R \phi^+ + \bar{e}_R \nu_{eL} \phi^- + \frac{1}{\sqrt{2}} (\bar{e} e H + i\, \bar{e} \gamma_5 e \zeta) \right] . \tag{5.2}$$

The first term, which provides the mass of electrons, thus owes its existence to spontaneous symmetry breaking. Since the Standard Model does not include a right handed component for the neutrino, a mass term for the neutrino is not generated by the Standard Model interactions.

But this is not completely true, as was first noted by Wolfenstein [74]. Interactions with a medium results in effective masses for the neutrinos belonging to different lepton families, leading to mixing and oscillations between the different neutrinos, an effect that has been used to explain the solar neutrino problem, as well as the shortfall of electron-antineutrinos coming from reactors. Neutrino oscillations occur because the mass eigenstates of the neutrinos are not identical with their flavour eigenstates. But the neutrino masses must be non-vanishing as well as close to one another for this argument to explain neutrino oscillations in vacuum. In material media the effective mass of the neutrino is significantly modified because of interactions. The change is different for the electron neutrino ν_e, which has both charged-current and neutral-current interactions with the electrons in the medium, compared to ν_μ and ν_τ which have only the second kind of interaction. This can enhance neutrino oscillations significantly, depending on the distance travelled in matter by the neutrinos.

Let us first see what happens to neutrinos propagating in vacuum [71, 75, 76]. If the neutrinos are all massless and thus degenerate eigenstates of the Hamiltonian, there will be no oscillation. Suppose however that the neutrinos have mass, different masses for different species, and further that the mass eigenstates are not identical with the flavour eigenstates. This assumption represents departure from the Standard Model. It is justified a posteriori by the observation of neutrino oscillations [77–81]. The price of this assumption is the introduction of additional dynamical degrees of freedom, at scales beyond current limits of experimental observation, to protect electroweak gauge symmetry. Then there will be mixing among neutrino eigenstates, which can be parametrized by a unitary matrix. The neutrino field ν_l which appears in a doublet with a lepton l is related to the field ν_α whose excitations are mass eigenstates by this matrix U as [82]

$$|\nu_{lL}\rangle = \sum_\alpha U_{l\alpha} |\nu_{\alpha L}\rangle \,. \tag{5.3}$$

At time t, the flavour eigenstates are related to the mass eigenstates by

$$|\nu_{lL}\rangle = \sum_\alpha e^{-iE_\alpha t} U_{l\alpha} |\nu_{\alpha L}\rangle \,. \tag{5.4}$$

It can be usually assumed quite safely that in any process different neutrinos are produced with spatial momenta of the same magnitude, so that if neutrinos are massless or if all neutrinos have the same mass, all the E_α are the same and there is no oscillation among the neutrino flavour states. I will then suppose that neutrinos have different masses, $m_\alpha \neq 0$ and all different. Then the probability of finding a $\nu_{l'}$ at time t in a beam that had started out as ν_l is given by

$$\begin{aligned} P_{\nu_{l'}\nu_l}(t) &= |\langle \nu_{l'} | \nu_l(t) \rangle|^2 \\ &= \sum_{\alpha,\beta} \left| U^*_{l'\alpha} U_{l\alpha} U^*_{l\beta} U_{l'\beta} \right| \cos\left((E_\alpha - E_\beta)t - \phi_{ll'\alpha\beta} \right) \,, \end{aligned} \tag{5.5}$$

where $\phi_{ll'\alpha\beta} = \arg\left(U^*_{l'\alpha}U_{l\alpha}U^*_{l\beta}U_{l'\beta}\right)$. The neutrinos are ultrarelativistic and start with the same spatial momenta, so I can write their energies as $E_\alpha \simeq E + \frac{m_\alpha^2}{2E}$. I can also replace the time of travel t by the distance of travel x and write

$$P_{\nu_{l'}\nu_l}(t) = \sum_{\alpha,\beta}\left|U^*_{l'\alpha}U_{l\alpha}U^*_{l\beta}U_{l'\beta}\right|\cos\left(\frac{(m_\alpha^2 - m_\beta^2)x}{2E} - \phi_{ll'\alpha\beta}\right) . \tag{5.6}$$

Clearly there will be no mixing and no oscillation if the neutrinos have vanishing mass in the vacuum. Interactions with a medium results in different effective masses for the neutrinos belonging to different lepton families, as first noted by Wolfenstein [74], but a neutrino mass is still needed. On a curved spacetime however, geometry provides an additional interaction with other fermions, thus a contribution to the Hamiltonian, challenging this conclusion.

Let us recall the nonlinear Dirac equation of Eq. (3.30)

$$\gamma^K e^\mu_K \partial_\mu\psi - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\gamma_K\sigma_{IJ}\psi + m\psi - \frac{i\kappa}{64}\bar{\psi}\{\gamma_K,\sigma_{IJ}\}\psi\{\gamma^K,\sigma^{IJ}\}\psi = 0 . \tag{5.7}$$

This equation was obtained from the total action of Eq. (3.19). The last term in the Dirac equation in particular, resulted from the on-shell expression of the spin connection given by Eq. (3.22b) i. e.,

$$A^{IJ}_\mu \equiv \omega^{IJ}_\mu[e] + \Lambda^{IJ}_\mu = \omega^{IJ}_\mu[e] + \frac{\kappa}{8}\bar{\psi}\{\gamma_K,\sigma^{IJ}\}\psi e^K_\mu . \tag{5.8}$$

Using the properties of γ and σ matrices (Appendix F), the Dirac equation can be written as

$$\gamma^K e^\mu_K \partial_\mu\psi - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\gamma_K\sigma_{IJ}\psi + m\psi + \frac{3i\kappa}{8}\left(\bar{\psi}\gamma^I\gamma^5\psi\right)\gamma_I\gamma^5\psi = 0 . \tag{5.9}$$

Also Einstein's equation is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} , \tag{5.10}$$

where the stress-energy tensor $T_{\mu\nu}$ is now quartic in the fermionic field,

$$\begin{aligned} T_{\mu\nu}(\psi,\bar{\psi}) = \frac{i}{4} & \left(\partial_\mu\bar{\psi}\gamma_I\psi e^I_\nu - \bar{\psi}\gamma_I\partial_\mu\psi e^I_\nu + \tfrac{i}{4}\omega^{IJ}_\mu e^K_\nu\bar{\psi}\{\gamma_K,\sigma_{IJ}\} + \psi + (\mu\leftrightarrow\nu)\right) \\ & + img_{\mu\nu}\bar{\psi}\psi - \tfrac{3\kappa}{16}g_{\mu\nu}\left(\bar{\psi}\gamma^I\gamma_5\psi\right)^2 . \end{aligned} \tag{5.11}$$

It should be noted that the above equations are for one species of fermion only. One important point often gets overlooked or at least is not explicitly mentioned, which is the fact that every fermion field must be included in the matter action and therefore all fermions will be present in the expression for spin connection,

$$A_\mu^{IJ} = \omega_\mu^{IJ} + \frac{\kappa}{8} e_\mu^K \sum_f \bar{\psi}_f \{\gamma_K, \sigma^{IJ}\} \psi_f \,, \tag{5.12}$$

where the sum is over all species of fermions present in the universe. This term will also appear in the nonlinear Dirac equation for each type of fermion,

$$\gamma^K e_K^\mu \partial_\mu \psi_i - \frac{i}{4} \omega_\mu^{IJ} e^{\mu K} \gamma_K \sigma_{IJ} \psi_i + m\psi_i + \frac{3i\kappa}{8} \left(\sum_f \bar{\psi}_f \gamma^I \gamma^5 \psi_f \right) \gamma_I \gamma^5 \psi_i = 0 \,. \tag{5.13}$$

We have seen in Eq. (5.8) that the spin connection comes out in the form $A_\mu^{IJ} = \omega_\mu^{IJ} + \Lambda_\mu^{IJ}$ on-shell. Let us see what happens if I decompose the spin connection in this form on-shell with Λ being independent of the tetrads as long as equations of motion are not used. With this decomposition I can write $F_{\mu\nu}^{IJ}(A)$ as

$$F_{\mu\nu}^{IJ}(A) = \widehat{F}_{\mu\nu}^{IJ}(\omega) + \partial_{[\mu} \Lambda_{\nu]}^{IJ} + \left[\omega_{[\mu}, \Lambda_{\nu]}\right] + \eta_{KL} \Lambda_{[\mu}^{IK} \Lambda_{\nu]}^{LJ} \,. \tag{5.14}$$

Extremising the action

$$\begin{aligned} S = \int |e| d^4x \quad & \Big[\tfrac{1}{2\kappa} \left(\widehat{F}_{\mu\nu}^{IJ}(\omega) + \partial_{[\mu} \Lambda_{\nu]}^{IJ} + \left[\omega_{[\mu}, \Lambda_{\nu]}\right] + \eta_{KL} \Lambda_{[\mu}^{IK} \Lambda_{\nu]}^{LJ} \right) e_I^\mu e_J^\nu \\ & + \tfrac{i}{2} \textstyle\sum_f \left(\bar{\psi}_f \gamma^K e_K^\mu \, \widehat{D}_\mu^f \psi_f - (\bar{\psi}_f \gamma^K e_K^\mu \, \widehat{D}_\mu^f \psi_f)^\dagger + 2m_f \bar{\psi}_f \psi_f \right) \\ & + \tfrac{1}{2\kappa} \eta_{KL} \Lambda_{[\mu}^{IK} \Lambda_{\nu]}^{LJ} e_I^\mu e_J^\nu + \tfrac{1}{8} \textstyle\sum_f e_K^\mu \Lambda_\mu^{IJ} \bar{\psi}_f \{\gamma^K, \sigma_{IJ}\} \psi_f \Big] \,. \end{aligned} \tag{5.15}$$

with respect to Λ I find that the only nonvanishing variations come from the fermionic part of the action and the last term of $F_{\mu\nu}^{IJ}(A)$, so that the equation of motion for Λ is

$$\Lambda_\mu^{IJ} = \frac{\kappa}{8} e_\mu^K \sum_f \bar{\psi}_f \{\gamma_K, \sigma^{IJ}\} \psi_f \,. \tag{5.16}$$

I can insert this solution for Λ into the Einstein's equations and the Dirac equation, which are then exactly the same as the equations found above. Furthermore, if I substitute this expression in the action, the resulting Einstein's equations and the Dirac equation are also exactly the same as found above. In general, inserting a solution into the action gives incorrect results. In this case however, the antisymmetrized covariant derivative of Λ contribute to a total derivative in the action, so Λ is an auxiliary field (see Appendix (C)).

The total action of gravity with fermions is thus

$$S = \int |e| d^4x \quad \Big[\tfrac{1}{2\kappa} \widehat{F}^{IJ}_{\mu\nu}(\omega) e^\mu_I e^\nu_J + \tfrac{i}{2} \textstyle\sum_f \Big(\bar{\psi}_f \gamma^K e^\mu_K \, \widehat{D}^f_\mu \psi_f - (\bar{\psi}_f \gamma^K e^\mu_K \, \widehat{D}^f_\mu \psi_f)^\dagger + 2 m_f \bar{\psi}_f \psi_f \Big) \\ + \tfrac{1}{2\kappa} \eta_{KL} \Lambda^{IK}_{[\mu} \Lambda^{LJ}_{\nu]} e^\mu_I e^\nu_J + \tfrac{1}{8} \textstyle\sum_f e^\mu_K \Lambda^{IJ}_\mu \bar{\psi}_f \{\gamma^K, \sigma_{IJ}\} \psi_f \Big] \, . \tag{5.17}$$

What I have is nothing more than general relativity with fermions. The contorsion Λ is an auxiliary field which enforces the interaction of spacetime geometry with fermionic fields but does not propagate. In the absence of fermions Λ vanishes, irrespective of any bosonic fields present as long as they are minimally coupled to gravity. Again this is all very well known, but writing the action in this form draws attention to another aspect which seems to have been overlooked.

The invariance of this action under local Lorentz transformations means that Λ transforms homogeneously under them. In particular, the last term of the above action is invariant on its own. Since Λ does not transform inhomogeneously, the coupling of Λ to fermions is not like the coupling of a gauge field to fermions. The transformation of fermions does not affect that of Λ, so their coupling is not protected by any invariance. This way it is more analogous to the coupling of a real scalar field to fermions – the coefficient of $\bar{\psi}\phi\psi$ can be freely set by hand. But unlike a scalar field, Λ can couple chirally to fermions – it couples to the left-handed neutrinos irrespective of whether or not there are right-handed neutrinos in the universe. So there is no reason why different species of fermions cannot be coupled to Λ with different coupling strengths, analogous to the Yukawa coupling of fermions to a scalar field.

Therefore the generic form of the action of fermions coupled to gravity must be, not (5.17), but

$$S = \int |e| d^4x \quad \Big[\tfrac{1}{2\kappa} \widehat{F}^{IJ}_{\mu\nu}(\omega) e^\mu_I e^\nu_J + \tfrac{i}{2} \textstyle\sum_f \Big(\bar{\psi}_f \gamma^K e^\mu_K \, \widehat{D}^f_\mu \psi_f - (\bar{\psi}_f \gamma^K e^\mu_K \, \widehat{D}^f_\mu \psi_f)^\dagger + 2 m_f \bar{\psi}_f \psi_f \Big) \\ + \tfrac{1}{2\kappa} \eta_{KL} \Lambda^{IK}_{[\mu} \Lambda^{LJ}_{\nu]} e^\mu_I e^\nu_J + \tfrac{1}{8} \textstyle\sum_f \Lambda^{IJ}_\mu e^\mu_K \big(\lambda_{fL} \bar{\psi}_{fL} \{\gamma^K, \sigma_{IJ}\} \psi_{fL} \\ + \lambda_{fR} \bar{\psi}_{fR} \{\gamma^K, \sigma_{IJ}\} \psi_{fR} \big) \Big] \, , \tag{5.18}$$

where I have taken into account the possibility that the tensor currents due to left and right-handed fermions, which transform independently under local Lorentz transformations, may couple to Λ with different coupling constants λ_{fL} and λ_{fR}, respectively. Even though in this form the action appears to be a philosophical departure from how fermions have always been treated in general relativity, it is in fact a generic form which must inevitably appear when fermions are put in curved spacetime, unless the coupling constants λ_f are set to zero by fiat. Furthermore, since Λ leads to a torsion

$$C^{\alpha}{}_{\mu\nu} \equiv \Lambda^{IJ}_{[\mu} e_{\nu]J} e^{\alpha}_{I} = \frac{\kappa}{2} \epsilon^{IJKL} e^{\alpha}_{I} e_{\mu J} e_{\nu K} \sum_{f} \lambda_f \bar{\psi}_f \gamma_L \gamma_5 \psi_f \,, \tag{5.19}$$

which is totally antisymmetric and thus does not affect geodesics, all particles fall at the same rate in a gravitational field and the principle of equivalence is not violated by these coupling constants. I have also checked that inclusion of coupling constants does not affect the current conservation of different fermions (Appendix E).

Solving for Λ and inserting the solution back into the action as before, produces the effective action

$$\begin{aligned} S = \int |e| d^4x \quad \Big[& \tfrac{1}{2\kappa} \widehat{F}^{IJ}_{\mu\nu}(\omega) e^{\mu}_{I} e^{\nu}_{J} + \tfrac{i}{2} \textstyle\sum_f \left(\bar{\psi}_f \gamma^K e^{\mu}_{K} \, \widehat{D}^{f}_{\mu} \psi_f - (\bar{\psi}_f \gamma^K e^{\mu}_{K} \, \widehat{D}^{f}_{\mu} \psi_f)^{\dagger} + 2 m_f \bar{\psi}_f \psi_f \right) \\ & - \tfrac{3\kappa}{16} \left(\textstyle\sum_f \left(\lambda_{fL} \bar{\psi}_{fL} \gamma_I \gamma^5 \psi_{fL} + \lambda_{fR} \bar{\psi}_{fR} \gamma_I \gamma^5 \psi_{fR} \right) \right)^2 \Big] \,. \end{aligned} \tag{5.20}$$

I will use this action of fermions in curved spacetime as the starting point of further calculations below. It is in fact not meaningful to work with a Dirac equation containing Λ, because Λ must always equal its on-shell value. Furthermore, the quartic term is independent of the background metric, but must be included as long as there is gravity in the universe. The only ways this term can be absent from the action are if gravity is turned off ($\kappa \to 0$), or if the quartic couplings λ_f are assumed to be zero. This term is suppressed by two powers of Planck mass compared to the mass term, but it could still help avert gravitational singularities [19, 65, 83, 84]. We will see that it can also in principle allow neutrino oscillations even when the neutrinos are massless.

5.1 NEUTRINO OSCILLATIONS

In considering the propagation of neutrinos through normal matter, i. e. solar or stellar cores or nuclear reactors, I need to take into account only the effects due to electrons and nucleons (or three colors each of up and down quarks) in addition to the quartic self-interaction of the neutrinos. Weak interactions will be present of course, I will come back to the effect of that. I will also restrict to only two types of neutrinos as before. The quartic term relevant to my purpose is

$$\begin{aligned}\mathscr{L}_{(\bar{\psi}\psi)^2} \quad = \quad & -\frac{3\kappa}{16}\left[\sum_{\alpha,\beta}\lambda_{\nu_\alpha}\lambda_{\nu_\beta}(\bar{\nu}_\alpha\gamma_I\nu_\alpha)(\bar{\nu}_\beta\gamma^I\nu_\beta)\right.\\ & \left. -2\sum_{\alpha,f}\lambda_{\nu_\alpha}(\bar{\nu}_\alpha\gamma_I\nu_\alpha)\left(-\lambda_{fV}\bar{\psi}_f\gamma^I\psi_f+\lambda_{fA}\bar{\psi}_f\gamma^I\gamma^5\psi_f\right)\right]+\cdots,\end{aligned} \tag{5.21}$$

where I have used the fact the neutrinos are left-handed, written $\lambda_V = \frac{1}{2}(\lambda_L - \lambda_R)$, $\lambda_A = \frac{1}{2}(\lambda_L + \lambda_R)$ for the other fermions, and indicated by dots the terms which do not involve neutrinos. It is easy to see that the ν_α which appear in the above expression, i. e. those which couple to Λ in (5.18), must be the mass eigenstates.

Following Wolfenstein [74] I calculate the forward scattering amplitude of the α-type neutrinos,

$$\mathcal{M} = -\frac{3\kappa}{8}\,(\bar{\nu}_\alpha\gamma_I\nu_\alpha)\,\lambda_{\nu_\alpha}\left\langle\sum_\beta\lambda_{\nu_\beta}\bar{\nu}_\beta\gamma^I\nu_\beta+\sum_{f=e,p,n}\left(\lambda_{fV}\bar{\psi}_f\gamma^I\psi_f-\lambda_{fA}\bar{\psi}_f\gamma^I\gamma^5\psi_f\right)\right\rangle, \tag{5.22}$$

where the average is taken over the background. In the second sum, the spatial components of the axial current average to spin in the nonrelativistic limit, which for normal matter is negligible. The axial charge is also negligible. Similarly, the spatial components of the vector current average to the spatial momentum of the background, which can also be neglected. Since neutrinos are ultrarelativistic, their density inside a finite volume such as a star is bounded by the rate of production times the average density of the region, i. e., several orders of magnitudes smaller than the density of electrons or baryons. Thus the average of the neutrino term can also be neglected.

So what I am left with is the average of the temporal component of the vector current of fermions, which is nothing but the number density of the fermions,

$$\langle \bar{\psi}\gamma^0\psi \rangle = -\langle \psi_f^\dagger \psi_f \rangle = -n_f\,. \tag{5.23}$$

It should be noted that the "density" of the fermion field is the time component of $J^\mu \equiv e^\mu_I \bar{\psi}\gamma^I\psi$. If the spacetime allows a 3+1 decomposition of the background metric as $g_{\mu\nu} = (-\lambda^2 + h_{ij})$, the volume measures can be related as $\sqrt{-g} = \lambda\sqrt{h}$ and $e^0_I = \lambda^{-1}\delta^0_I$, where δ^μ_I is the Kronecker delta. In this case $J^0 = -\lambda^{-1}\psi^\dagger\psi$ which is integrated over three spatial dimensions against the volume measure $\lambda\sqrt{h}$. I have assumed this decomposition.

The contribution of the forward scattering amplitude to the effective Hamiltonian density is therefore

$$\delta\mathscr{H}_{\text{eff}} = \left(\sum_{f=e,p,n} \lambda_f n_f\right) \sum_\alpha \lambda_{\nu_\alpha} \nu_\alpha^\dagger \nu_\alpha, \tag{5.24}$$

where I have now dropped the subscript V and absorbed a factor of $\sqrt{\frac{3\kappa}{8}}$ in the definition of each of the λ.

This term acts as an effective mass term for the neutrinos, with $m_\alpha = \lambda_{\nu_\alpha}\rho$, where $\rho = \sum \lambda_f n_f$ is a weighted density of fermions (excluding neutrinos) that is the same for all neutrinos. This effective mass term modifies the mass of the neutrino and thus the oscillation formula, but even more interestingly, this term will cause neutrino oscillations even if neutrinos are massless. In that case, with two species of neutrinos I should replace $|m_2^2 - m_1^2|$ by $\rho^2|\lambda_{\nu_2}^2 - \lambda_{\nu_1}^2|$ for constant density. The mixing matrix takes the form $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, so the probability of conversion of one particular flavour of neutrino into the other in a regions with constant density becomes

$$P_{\text{conv}} = \sin^2 2\theta \sin^2\left(\frac{\rho^2\Delta\lambda^2}{4E}x\right)\,, \tag{5.25}$$

where $\Delta\lambda^2 = |\lambda_{\nu_2}^2 - \lambda_{\nu_1}^2|$.

This result is qualitatively different from the usual formula for neutrino oscillations in vacuum. If I do not write a mass term for the neutrino, all contributions to neutrino mass comes from the quartic interaction of the neutrino with fermions in the background as well as with itself. The actual background geometry of the spacetime does not contribute to the effective mass, at least for small curvatures, for which the leading order result of the forward scattering amplitude is sufficient. Thus a neutrino propagating through vacuum would not oscillate into different flavours, but oscillation would occur only in the region where there is a fermion density and stop when the neutrino leaves that region. This is exactly like what happens for oscillation due to weak interactions, except for the fact that in the case of quartic interaction induced by torsion, leptons and baryons all contribute to the effective mass of neutrinos. It should be noted that the coupling constants λ cannot be fixed by appealing to a more fundamental theory, but are in principle measurable by looking at oscillations when the neutrinos pass through different media, such as stars with different baryon densities, or nuclear reactor cores.

A non-vanishing λ_V for any fermion requires that the left-handed component of the fermion does not couple to torsion with the same strength as the right-handed component. Thus chiral symmetry is broken by torsion, or alternatively, by the quartic term which has its origin in spacetime geometry. It should be noted that it is not only neutrinos, but all fermions get a contribution to their masses from this geometrical mechanism. Even if I assume that the contribution to effective mass is of the same order for all fermions in the same background matter density, the mass of very dense stars can be significantly larger than what is calculated from their baryon count. This can be expected to affect stellar models, dark matter estimates, and cosmology.

5.2 WEAK INTERACTIONS

Neutrinos passing through matter will also interact with it via electroweak gauge fields. In this case, if I look at the effective four-fermion interaction at lowest order, only the interactions with electrons are relevant. This is because the weak interaction couples flavour eigenstates of the neutrinos with other fields; ν_e couples to electrons via both charged and neutral currents, while ν_μ couples to electrons only via the neutral current. The modification of the mixing angle due to weak interactions in normal matter is straightforward to calculate [82], as I will show in outline below. The effective Lagrangian due to the charged current interaction can be written as

$$\mathscr{L}_{\mathrm{cc}} = -\frac{G_F}{\sqrt{2}} \left(\bar{\psi}_e \gamma^I (1-\gamma^5)\psi_e\right) \left(\bar{\nu}_e \gamma_I (1-\gamma^5)\nu_e\right) \,, \tag{5.26}$$

where a Fierz identity has been used (see Appendix F). The (elastic) forward scattering amplitude provides the contribution to the Hamiltonian, $\sqrt{2}G_F \langle \bar{\psi}_e \gamma^I (1-\gamma^5)\psi_e \rangle (\bar{\nu}_{eL}\gamma_I \nu_{eL}) \simeq \sqrt{2}G_F n_e \nu^{\dagger}_{eL}\nu_{eL}$. Normal matter does not contain muons, so ν_μ does not have a charged current interaction.

Both flavours of neutrinos have the same neutral current interactions, so that the contribution appears as a common term to the Hamiltonian,

$$V_{\rm nc} = \sqrt{2}G_F \sum_{f=e,p,n} n_f \left[I^f_{3L} - 2\sin^2\theta_W Q^f \right] , \tag{5.27}$$

where I^f_{3L} is the third component of weak isospin for the left-handed component of the fermion f and Q_f is its charge. For electrically neutral normal matter, the electron and proton contributions cancel each other and I am left with only the neutron contribution, equal to $-\sqrt{2}G_F n_n/2$ for both types of neutrinos. The Hamiltonian, diagonal in the space of mass eigenstates, can thus be written in flavour space as

$$H = H_c \mathbb{I} + \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2}G_F n_e & 0 \\ 0 & 0 \end{pmatrix} . \tag{5.28}$$

Here I have written H_c for the common terms in the Hamiltonian, and $\Delta m^2 = \rho^2 \left| \lambda^2_{\nu_2} - \lambda^2_{\nu_1} \right|$. The effective mixing angle $\tilde{\theta}$, including the effects of both the geometric and weak contributions, for constant densities is thus given by

$$\tan 2\tilde{\theta} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F n_e E} . \tag{5.29}$$

This formula is for ultrarelativistic neutrinos, and thus valid only in regions where matter density is not too high. For regions with low matter density and $n_e \simeq n_p \geq n_n$ and $n_e \to 0$, I find that the right hand side is proportional to n_e/E . For three generations of leptons I can make similar substitutions into the standard formula for neutrino oscillations. For neutrinos passing through regions where the matter density is not constant (MSW effect [74, 85, 86]), nonlinearity introduces additional complications particularly for very large matter densities, since effective masses of neutrinos and thus Δm^2 , can vary greatly in such situations. I will not attempt to do that calculation here.

If I am interested only in calculating neutrino oscillations, I could take a pragmatic approach and start with Eq. (5.21) as the defining interaction term. This term is very similar to what is called non-standard neutrino interactions (NSI) [87–90], in this case flavour-changing in the neutrino sector. However, the geometrical origin of this interaction means that all fermions are in quartic interaction with one another, including themselves. At low energies and for matter at normal densities, the only effect of this is expected to be on neutrino dynamics as I have discussed in this chapter, but at high energies as well as for high densities of matter, for example in stellar collapse or in the early universe, I can expect this interaction to play an important role. It is also not meaningful to talk about the quartic interactions in the absence of gravity. This is related to the fact that the quartic term appears to make the model nonrenormalizable by power counting. Because of their origin from curved spacetime, the quartic couplings contain in them a factor of $\sqrt{\kappa}$ and thus must vanish in the flat space limit. So the counterterms in curved spacetime will have to involve curvature, thus the question of renormalizability cannot be addressed without a theory of quantum gravity, as has been noted elsewhere [91].

The second point is about the size of the quartic term. Is the contribution of this term to neutrino oscillations negligibly small? I think that this question cannot be answered purely theoretically. Unlike in the case of weak interactions, where the energy required to create W-boson pairs from the vacuum sets the scale of the four-fermion interaction (and the oscillation formula can be calculated directly from quantum field theory [92]), here the scale is not related to the quantum dynamics of the contorsion Λ, which does not in fact have any dynamics. Therefore the coupling constants λ are free and can be set only by comparison with experimental data, not from any theoretical argument. By comparing with the NSI couplings, I can expect that the λ are one or two orders of magnitude smaller than the effective quartic couplings coming from weak interactions, i. e., than the Fermi constant. If the neutrinos are massless in vacuum, the flavour-changing interaction becomes crucial for oscillations inside matter, even if it is small.

It should be noted that the use of torsion for oscillation of massless neutrinos has been proposed earlier in [93]. A coupling of neutrinos to torsion analogous to the last term in Eq. (5.17) was proposed, with different couplings for different species of neutrinos. In this case the torsion is proportional to the spin density of the background, which for normal matter – i. e. if spins are not aligned – averages to zero over macroscopic volumes, so the effect on oscillations is very small. By breaking chiral symmetry in the coupling of fermions to torsion, and by using the fact that all fermions couple to torsion, I expect to find a much larger effect. There have been proposals of nonuniversal gravitational couplings of neutrinos leading to oscillations [94, 95], with the nonuniversality of couplings being subject to experimental constraints. In this case the equivalence principle is violated at the quantum level. In our proposal, nonuniversality of fermion couplings is restricted to their couplings with torsion, while their coupling with background gravity is universal – all particles continue to fall at the same rate.

6

PERTURBATIONS IN VIERBEIN-EINSTEIN-PALATINI FORMALISM

General relativity is a nonlinear theory. A linearised approximation to the theory can be obtained by considering the weak field limit of gravitational field. The spacetime metric can be considered as a flat background metric (Minkowski) plus a perturbation [17]. Einstein's equation turns out to be a linear second order equation in the perturbation. In this chapter[1] I will consider the weak field limit of the vierbein-Einstein-palatini variables. The idea of perturbing the vierbein is not a new one (a representative list is [96–104]), but our work differs from earlier work in an important manner. In all papers dealing with vierbein perturbations that I have been able to find, the perturbations were considered around a flat background spacetime. In such cases, the background spacetime and the internal Lorentz space are identical, and as a result the background tetrad can be written as a Kronecker delta , $e^I_\mu = \delta^I_\mu$. Furthermore, the background connection is then trivially flat and torsion-free, which means no attention is paid to the independent nature of the spin connection. In this chapter I will assume a general curved background and an independent spin connection, so tetrad fields are not constant, while the absence of torsion is implemented by the equations of motion for the spin connection only in the absence of fermionic matter. When fermions are coupled to the spin connection, torsion remains as a part of the spin connection, and the latter cannot be completely eliminated from the perturbation equations. Gravity with torsion has seen interests in many places [101, 105–108]. I am interested in perturbation of tetrads and spin connection in particular, as torsion can be written in terms of these variables.

1 The work reported here is based on the paper "Weak field limit in vierbein-Einstein-Palatini formalism and Fierz-Pauli Equation", Subhasish Chakrabarty and Amitabha Lahiri, arXiv:1507.03884 [gr-qc].

I will briefly recapitulate the formalism of metric perturbation theory. In this, the spacetime metric is written as a background metric plus a perturbation. Often we are interested in a flat background, in which case

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{6.1}$$

with

$$|h_{\mu\nu}| \ll 1 \tag{6.2}$$

for each component. It should be noted that $h_{\mu\nu}$ is often defined with a factor of G or κ so that its smallness as given by the above equation becomes apparent. The inverse is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,, \tag{6.3}$$

where all indices are raised and lowered with η .

The linearised Riemann and Ricci tensors, and the Ricci scalar, are thus

$$R_{\mu\nu\rho\sigma} = \partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\,, \tag{6.4}$$

$$R_{\mu\nu} = \frac{1}{2}\left[\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right]\,, \tag{6.5}$$

$$R = \partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\,, \tag{6.6}$$

where h is the trace of the perturbation, $h = \eta^{\mu\nu}h_{\mu\nu}$. The linearised Einstein's equation in vacuum follows from these,

$$\frac{1}{2}[\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h] = \kappa T_{\mu\nu}\,, \tag{6.7}$$

which is the linearised form of $G_{\mu\nu} = \kappa T_{\mu\nu}$. The linearised version written above is for a special situation where the background is taken to be flat. I will consider perturbations around a general curved background. I will denote the background metric by $\bar{g}_{\mu\nu}$ so that the total metric is written as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Then the Christoffel symbols are

$$\widehat{\Gamma}^\alpha{}_{\mu\nu} = \bar{\widehat{\Gamma}}^\alpha{}_{\mu\nu} - \frac{1}{2}h^{\alpha\lambda}\left[\partial_\mu\bar{g}_{\lambda\nu} + \partial_\nu\bar{g}_{\mu\lambda} - \partial_\lambda\bar{g}_{\mu\nu}\right] + \frac{1}{2}\bar{g}^{\alpha\lambda}\left[\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right] + \mathcal{O}(h^2), \tag{6.8}$$

where now the indices are lowered and raised by $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$, respectively, and quantities pertaining to the background are denoted by a bar.

By calculating the Einstein tensor using these Christoffel symbols, I can write the Einstein's equation for an arbitrary background,

$$\bar{R}_{\mu\nu} - \frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta} + h_{\mu\nu}\bar{g}^{\alpha\beta} - \bar{g}_{\mu\nu}h^{\alpha\beta}\right)\bar{R}_{\alpha\beta} + R^1_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}R^1_{\alpha\beta} = 8\pi G T_{\mu\nu}\,, \tag{6.9}$$

where the Ricci tensor $\bar{R}_{\mu\nu}$ is derived from the background metric $\bar{g}_{\mu\nu}$, and the quantity $R^1_{\mu\nu}$ is the part of the Ricci tensor linear in h. This equation, however, contains the background equation $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 8\pi G\bar{T}_{\mu\nu}$ which needs to be eliminated. It should be noted that if the stress-energy tensor is calculated from a matter action by varying the metric, $T_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ are not equal in general. Thus I can finally write the equation for gravitational perturbations as

$$\frac{1}{2}\left(\bar{g}_{\mu\nu}h^{\alpha\beta} - h_{\mu\nu}\bar{g}^{\alpha\beta}\right)\bar{R}_{\alpha\beta} + R^1_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}R^1_{\alpha\beta} = 8\pi G\left(T_{\mu\nu} - \bar{T}_{\mu\nu}\right)\,. \tag{6.10}$$

In this chapter, I will consider the weak field limit of the vierbein-Einstein-Palatini variables by perturbing them around arbitrary background. The perturbation of the tetrads can be related to metric perturbation in General Relativity. For the spin conneciton, as long as it is independent, the perturbation should be treated as an independent quantity as well. Although the spin connection is expressible in terms of the tetrad in the absence of matter, in particular fermionic matter, as Eq. (2.28) relies crucially on the connection being torsion-free, which in this formalism follows from the matter action being independent of the spin connection A_μ^{IJ}. But in general one should treat the tetrads and spin connection as independent variables. Thus, perturbations of these variables around some background solution of the equations of motion should be considered as independent objects. This is because perturbations are off-shell objects a priori and need not satisfy the same equations as the background solutions.

For the present purpose I will consider the perturbation of the co-tetrad first and write e_I^μ as a sum of the background and perturbation,

$$e_I^\mu = \bar{e}_I^\mu + f_I^\mu, \tag{6.11}$$

where f_I^μ is much smaller than $\bar{e}_I^\mu$ (more precisely, $\mathrm{Tr}(\bar{e}^I_\mu f_I^\mu) \ll 1$). I will denote background quantities by a bar on top. In order to calculate the perturbation of tetrads, I use their definition $e^\mu_I e^I_\nu = \delta^\mu_\nu = \bar{e}^\mu_I \bar{e}^I_\nu$, where the internal index is raised and lowered with η_{IJ} as before. Thus I find

$$e^I_\mu = \bar{e}^I_\mu - \bar{e}^J_\mu \bar{e}^I_\alpha f^\alpha_J . \tag{6.12}$$

I will often denote $-\bar{e}^J_\mu \bar{e}^I_\alpha f^\alpha_J$ as $\widetilde{f}^I_\mu$. The spacetime indices will be raised and lowered by the total spacetime metric $g_{\mu\nu} = e^I_\mu e_{I\nu}$ when needed. By writing the background metric as $\bar{g}_{\mu\nu} = \bar{e}^I_\mu \bar{e}_{I\nu}$, I can identify the metric perturbation $h_{\mu\nu}$ in terms of the background tetrad $\bar{e}^I_\mu$ and the tetrad perturbation $\widetilde{f}^I_\mu$ as

$$h_{\mu\nu} = \bar{e}_{I\mu} \widetilde{f}^I_\nu + \bar{e}_{I\nu} \widetilde{f}^I_\mu . \tag{6.13}$$

The background now is any general spacetime, not necessarily flat. I will also write the spin connection as a sum of its value in the background spacetime and a perturbation,

$$A^{IJ}_\mu = \bar{A}^{IJ}_\mu + a^{IJ}_\mu . \tag{6.14}$$

However, since all components $\bar{A}^{IJ}_\mu$ of the background spin connection may vanish, it is not sensible to treat a^{IJ}_μ as small perturbation. In particular, I will not neglect terms quadratic in a^{IJ}_μ when calculating the action. Thus I calculate the affine connection up to first order in the perturbation f as

$$\Gamma^\alpha{}_{\mu\nu} = \bar{\Gamma}^\alpha{}_{\mu\nu} + \bar{e}^\alpha_I \partial_\mu \widetilde{f}^I_\nu + f^\alpha_I \partial_\mu \bar{e}^I_\nu + \bar{A}^{IJ}_\mu \bar{e}^J_\nu f^\alpha_I + a^{IJ}_\mu \bar{e}^J_\nu f^\alpha_I + a^{IJ}_\mu \bar{e}^\alpha_I \widetilde{f}^J_\nu . \tag{6.15}$$

Here $\bar{\Gamma}$ corresponds to the background spacetime,

$$\bar{\Gamma}^\alpha{}_{\mu\nu} = \bar{e}^\alpha_I \partial_\mu \bar{e}^I_\nu + \bar{A}^I_{\mu J} \bar{e}^J_\nu \bar{e}^\alpha_I . \tag{6.16}$$

I will also write the curvature in terms of $\bar{A}$ and a,

$$F^{IJ}_{\mu\nu} = \bar{F}^{IJ}_{\mu\nu} + \mathcal{F}^{IJ}_{\mu\nu} , \tag{6.17}$$

where $\bar{F}$ is the background curvature and $\mathcal{F}$, the extra part due to perturbation, can be written in the form

$$\mathcal{F}^{IJ}_{\mu\nu} = \bar{D}_\mu a^{IJ}_\nu - \bar{D}_\nu a^{IJ}_\mu + [a_\mu, a_\nu]^{IJ} , \tag{6.18}$$

with $\bar{D}$ being the covariant derivative corresponding to the background spin connection $\bar{A}$. The perturbed vierbein-Einstein-Palatini action is thus

$$S = \frac{1}{16\pi G}\int |e|\, d^4x \left[F^{IJ}_{\mu\nu}\bar{e}^{\mu}_{I}\bar{e}^{\nu}_{J} + 2F^{IJ}_{\mu\nu}\bar{e}^{\mu}_{I}f^{\nu}_{J} + F^{IJ}_{\mu\nu}f^{\mu}_{I}f^{\nu}_{J}\right] + S_M\,. \tag{6.19}$$

The determinant $|e| \equiv |\bar{e}^I_\mu + \widetilde{f}^I_\mu|$ is now a polynomial in f. The lowest order field equations will be of first order in f. The variation of the determinant produces

$$\delta|e| = -|e|(\bar{e}^K_\alpha + \widetilde{f}^K_\alpha)\delta f^\alpha_K\,, \tag{6.20}$$

using which I obtain field equations by varying the vierbein-Einstein-Palatini action of Eq. (6.19),

$$(\bar{F}^{IJ}_{\alpha\mu} + \mathcal{F}^{IJ}_{\alpha\mu})(\bar{e}^\alpha_I + f^\alpha_I) - \frac{1}{2}(\bar{e}^J_\mu + \widetilde{f}^J_\mu)(\bar{F}^{IJ}_{\alpha\beta} + \mathcal{F}^{IJ}_{\alpha\beta})(\bar{e}^\alpha_K + f^\alpha_K)(\bar{e}^\beta_L + f^\beta_L) = 8\pi G T_{\mu\nu}e^{\nu J}\,. \tag{6.21}$$

Of course, I could have obtained these directly from Eq. (2.38) (neglecting $\mathcal{O}(\kappa^2)$ terms) by replacing $e \to \bar{e} + \widetilde{f}$ and $A \to \bar{A} + a$. However, it is useful to construct the action for the perturbations, as I will later consider matter couplings.

Subtracting the vierbein-Einstein-Palatini equation for the background, I get the equation of motion for the vierbein-Einstein-Palatini perturbations,

$$\begin{aligned}\bar{F}^{IJ}_{\alpha\mu}f^\alpha_I - \frac{1}{2}\bar{F}^{KL}_{\alpha\beta}\left[\widetilde{f}^J_\mu\bar{e}^\alpha_K\bar{e}^\beta_L + 2\bar{e}^J_\mu f^\alpha_K\bar{e}^\beta_L\right] + \mathcal{F}^{IJ}_{\alpha\mu}\left(\bar{e}^\alpha_I + f^\alpha_I\right)\\ -\frac{1}{2}\mathcal{F}^{KL}_{\alpha\beta}\left[\bar{e}^J_\mu\bar{e}^\alpha_K\bar{e}^\beta_L + \widetilde{f}^J_\mu\bar{e}^\alpha_K\bar{e}^\beta_L + 2\bar{e}^J_\mu f^\alpha_K\bar{e}^\beta_L\right] = 8\pi G\left(T_{\mu\nu}e^{\nu J} - \bar{T}_{\mu\nu}\bar{e}^{\nu J}\right)\,.\end{aligned} \tag{6.22}$$

This is a generic equation in the sense that I have not considered any particular background, or required that the background be flat, so this is the equation of perturbations around a general background spacetime.

7

CONCLUSION AND SUMMARY

I will conclude the thesis with chapter-wise summary of the calculations and results of different chapters.

- Summary of chapter 2:

  In chapter 2 I have introduced the tetrads and the spin connection which are the basic variables in the vierbein-Einstein-Palatini formalism. I have shown how torsion is non-zero at the level of action. I have written the Riemann tensor, Ricci tensor and Ricci scalar in terms of the tetrads and the spin connection and eventually defined the action. We have seen that in case of vacuum or minimally coupled bosonic field, the vierbein-Einstein-Palatini formalism is equivalent to General Relativity. I have also discussed the torsion-free tetrad formulation of General Relativity.

- Summary of chapter 3:

  In chapter 3 I have considered different matter fields in the vierbein-Einstein-Palatini formalism. For scalar field and electromagnetic field, I have shown that the torsion-free condition can be imposed a priori. I have considered fermionic field in the tetrad formulation as well as the vierbein-Einstein-Palatini formalism. We have seen that the Dirac equation and Einstein's equation get contributions due to the spin-torsion coupling. I have shown that torsion does not affect the conservation of current of the fermionic field.

- Summary of chapter 4:

  In chapter 4 I have discussed conformal transformations in terms of the tetrads and the spin connection. I have shown that at the level of action, the transformation of the spin connection remains indeterminate and one can make different choices of its transformation. I have considered two such choices and shown how they affect the dynamics of different fields. For dynamically generated or on-shell torsion, I have shown that it breaks the conformal invariance of the matter field equations. I have written a general conformal transformation of the off-shell spin connection.

- Summary of chapter 5:

  In chapter 5 we have seen how the four-fermion interaction resulting from spin-torsion coupling leads to effective masses for neutrinos. I have shown that the Standard Model cannot explain the mass generation for neutrinos although mass is necessary to explain neutrino oscillations. I have written a chiral symmetry breaking Lagrangian for fermions and shown how it can potentially give masses to neutrinos. I have also briefly discussed neutrino oscillations due to weak interactions.

- Summary of chapter 6:

  In chapter 6 I have considered the perturbations of the tetrads and the spin connections. Starting with the weak field limit in General Relativity where the background is taken to be flat, I have gone to perturbations on non-flat background. I have written the most general equation of perturbations in terms of the vierbein-Einstein-Palatini variables.

Overall, this thesis is based on a formalism of gravity that looks like other gauge theories. This formalism not only allows one to consider fermions in curved spacetime, but also gives new insights. One of the reasons why torsion is usually neglected is that its effects are small compared to the other terms in field equations. But we have seen that torsion can potentially lead the effective mass generation of neutrinos in regions of fermion density. The contribution in the mass term comes from all fermions including leptons and baryons, and it affects not only neutrinos but all the fermions. In normal matter densities this effect can be negligible for heavier fermions, but in extreme objects it could lead to observable effects and thus open up interesting possibilities of future works.

Appendices

A

TORSION, CURVATURE AND BIANCHI IDENTITIES

A.1 CONTORSION TENSOR

The covariant derivatives on a vector v and a covector w are defined respectively as

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda} v^\lambda \,, \tag{A.1}$$

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\lambda{}_{\mu\nu} w_\lambda \,. \tag{A.2}$$

I will find the expression for the affine connection in terms of the Christoffel symbols and contorsion tensor using metric compatibility i. e.,

$$0 = \nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda} \,. \tag{A.3}$$

Cyclic permutation in the indices α, μ, ν gives

$$\partial_\mu g_{\nu\alpha} - \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} - \Gamma^\lambda{}_{\mu\alpha} g_{\nu\lambda} = 0 \,, \tag{A.4}$$

$$\partial_\nu g_{\alpha\mu} - \Gamma^\lambda{}_{\nu\alpha} g_{\lambda\mu} - \Gamma^\lambda{}_{\nu\mu} g_{\alpha\lambda} = 0 \,. \tag{A.5}$$

Now (A.4) + (A.5) − (A.3) gives

$$\begin{aligned}(\Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{\nu\mu}) g_{\alpha\lambda} + (\Gamma^\lambda{}_{\nu\alpha} - \Gamma^\lambda{}_{\alpha\nu}) g_{\mu\lambda} + (\Gamma^\lambda{}_{\mu\alpha} - \Gamma^\lambda{}_{\alpha\mu}) g_{\nu\lambda} &= \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} \\ &\quad - \partial_\alpha g_{\mu\nu} \\ \Rightarrow (\Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{\nu\mu}) g_{\alpha\lambda} + C^\lambda{}_{\nu\alpha} g_{\mu\lambda} + C^\lambda{}_{\mu\alpha} g_{\nu\lambda} &= \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \,,\end{aligned} \tag{A.6}$$

where C is the torsion tensor defined by Eq. (1.6). I will break the affine connection in symmetric and antisymmetric parts in the last two indices,

$$
\begin{aligned}
& \Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}(\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\nu\mu}) + \frac{1}{2}C^{\alpha}{}_{\mu\nu} \\
\Rightarrow \quad & \Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\nu\mu} = 2\Gamma^{\alpha}{}_{\mu\nu} - C^{\alpha}{}_{\mu\nu} \,.
\end{aligned} \tag{A.7}
$$

It should be noted that the symmetric part is not only the Christoffel symbols because although the torsion tensor is antisymmetric in the last two indices, I can always construct a symmetric combination in the first two indices. Using the above I replace the symmetric part in Eq. (A.6) and get

$$
\Gamma^{\alpha}{}_{\mu\nu} = \widehat{\Gamma}^{\alpha}{}_{\mu\nu} - \frac{1}{2}(C_{\mu\nu}{}^{\alpha} + C_{\nu\mu}{}^{\alpha} - C^{\alpha}{}_{\mu\nu}) \,. \tag{A.8}
$$

Here $\widehat{\Gamma}^{\alpha}{}_{\mu\nu}$ are the Christoffel symbols given by Eq. (1.3) and $\frac{1}{2}(C_{\mu\nu}{}^{\alpha} + C_{\nu\mu}{}^{\alpha} - C^{\alpha}{}_{\mu\nu}) \equiv S^{\alpha}{}_{\mu\nu}$ is the contorsion tensor. Clearly the first two terms in $S^{\alpha}{}_{\mu\nu}$ are symmetric in $\mu\nu$. Also, the contorsion tensor is antisymmetric in the first and the last indices.

A.2 RIEMANN CURVATURE TENSOR

In General Relativity the Riemann tensor is defined by

$$
(\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu} - \widehat{\nabla}_{\nu}\widehat{\nabla}_{\mu})v^{\rho} = \widehat{R}^{\rho}{}_{\sigma\mu\nu}v^{\sigma} \,. \tag{A.9}
$$

I will find the expression of the above commutator for a general affine connection with non-zero torsion. I will find this for vector, covector and tensor respectively.

$$
\begin{aligned}
(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})v^{\rho} &= \partial_{\mu}\nabla_{\nu}v^{\rho} - \Gamma^{\sigma}{}_{\mu\nu}\nabla_{\sigma}v^{\rho} + \Gamma^{\rho}{}_{\mu\sigma}\nabla_{\nu}v^{\sigma} - (\mu \leftrightarrow \nu) \\
&= \partial_{\mu}\partial_{\nu}v^{\rho} + (\partial_{\mu}\Gamma^{\rho}{}_{\nu\lambda})v^{\lambda} + \Gamma^{\rho}{}_{\nu\lambda}\partial_{\mu}v^{\lambda} - \Gamma^{\sigma}{}_{\mu\nu}\nabla_{\sigma}v^{\rho} + \Gamma^{\rho}{}_{\mu\sigma}\partial_{\nu}v^{\sigma} \\
&\qquad + \Gamma^{\rho}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\lambda}v^{\lambda} - (\mu \leftrightarrow \nu) \\
&= R^{\rho}{}_{\sigma\mu\nu}v^{\sigma} - C^{\sigma}{}_{\mu\nu}\nabla_{\sigma}v^{\rho} \,,
\end{aligned} \tag{A.10}
$$

where

$$
R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\mu\sigma} \,. \tag{A.11}
$$

Proceeding in the same way as above for a covector w_σ I get

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) w_\sigma = -R^\rho{}_{\sigma\mu\nu} w_\rho - C^\rho{}_{\mu\nu} \nabla_\rho w_\sigma \,. \tag{A.12}$$

For a general tensor $T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots}$,

$$\begin{aligned}(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots} = R^{\alpha_1}{}_{\alpha_1'\mu\nu} T^{\alpha_1'\alpha_2\ldots}{}_{\beta_1\beta_2\ldots} + R^{\alpha_2}{}_{\alpha_2'\mu\nu} T^{\alpha_1\alpha_2'\ldots}{}_{\beta_1\beta_2\ldots} + \cdots \\ - R^{\beta_1'}{}_{\beta_1\mu\nu} T^{\alpha_1\alpha_2\ldots}{}_{\beta_1'\beta_2\ldots} - R^{\beta_2'}{}_{\beta_2\mu\nu} T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2'\ldots} \\ - C^\rho{}_{\mu\nu} \nabla_\rho T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots} - \cdots \,.\end{aligned} \tag{A.13}$$

A.3 SYMMETRIES AND BIANCHI IDENTITIES

A.3.1 *Skew symmetry*

Riemann tensor is antisymmetric in the last pair of indices

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} \,, \tag{A.14}$$

by definition. I will find out the same in the first pair of indices. For this purpose I will use Eq. (A.13) for metric tensor. Using metric compatibility,

$$\begin{aligned} 0 &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) g_{\alpha\beta} = -R^\rho{}_{\alpha\mu\nu} g_{\rho\beta} - R^\rho{}_{\beta\mu\nu} g_{\alpha\rho} \\ &\Rightarrow R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \,. \end{aligned} \tag{A.15}$$

So skew symmetry is same in General Relativity and torsion gravity.

A.3.2 *First Bianchi identity*

Using the expression for Riemann tensor from Eq. (A.11), I have

$$
\begin{aligned}
R^{\rho}{}_{[\sigma\mu\nu]} &= \partial_{[\mu}\Gamma^{\rho}_{\sigma\nu]} - \partial_{[\nu}\Gamma^{\rho}_{\sigma\mu]} + \Gamma^{\rho}_{[\mu|\alpha|}\Gamma^{\alpha}_{\nu\sigma]} - \Gamma^{\rho}_{[\nu|\alpha|}\Gamma^{\alpha}_{\mu\sigma]} \\
&= \partial_{[\mu}C^{\rho}{}_{\nu\sigma} + \Gamma^{\rho}{}_{[\mu|\lambda|}C^{\lambda}{}_{\nu\sigma]} \\
&= \nabla_{[\sigma}C^{\rho}{}_{\mu\nu]} + C^{\rho}{}_{\lambda[\sigma}C^{\lambda}{}_{\mu\nu]} \,. \qquad \text{(A.16)}
\end{aligned}
$$

A.3.3 *Pairwise symmetry*

In General Relativity Riemann tensor is symmetric under the interchange of first and last pair of indices. I will use skew symmetry and first Bianchi identity to derive the difference between $R_{\rho\sigma\mu\nu}$ and $R_{\mu\nu\rho\sigma}$. I have

$$
\begin{aligned}
R_{\rho\sigma\mu\nu} - R_{\mu\nu\rho\sigma} =\; & R_{\rho\sigma\mu\nu} - (\nabla_{[\nu}C_{|\mu|\rho\sigma]} + C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]} - R_{\mu\rho\sigma\nu} - R_{\mu\sigma\nu\rho}) \\
=\; & R_{\rho\sigma\mu\nu} - R_{\rho\mu\sigma\nu} - R_{\sigma\mu\nu\rho} - \nabla_{[\nu}C_{|\mu|\rho\sigma]} - C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]} \\
=\; & R_{\rho\sigma\mu\nu} - (\nabla_{[\mu}C_{|\rho|\sigma\nu]} + C_{\rho\lambda[\mu}C^{\lambda}{}_{\sigma\nu]} - R_{\rho\sigma\nu\mu} - R_{\rho\nu\mu\sigma}) \\
& - (\nabla_{[\mu}C_{|\sigma|\nu\rho]} + C_{\sigma\lambda[\mu}C^{\lambda}{}_{\nu\rho]} - R_{\sigma\nu\rho\mu} - R_{\sigma\rho\mu\nu}) \\
& - \nabla_{[\nu}C_{|\mu|\rho\sigma]} - C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]} \\
=\; & R_{\sigma\nu\rho\mu} - (R_{\rho\nu\sigma\mu} + R_{\rho\sigma\mu\nu}) - \nabla_{[\mu}C_{|\rho|\sigma\nu]} + C_{\rho\lambda[\mu}C^{\lambda}{}_{\sigma\nu]} \\
& - \nabla_{[\mu}C_{|\sigma|\nu\rho]} - C_{\sigma\lambda[\mu}C^{\lambda}{}_{\nu\rho]} - \nabla_{[\nu}C_{|\mu|\rho\sigma]} - C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]} \\
=\; & R_{\sigma\nu\rho\mu} - R_{\rho\mu\sigma\nu} - \nabla_{[\mu}C_{|\sigma|\nu\rho]} - C_{\sigma\lambda[\mu}C^{\lambda}{}_{\nu\rho]} \\
& - \nabla_{[\nu}C_{|\mu|\rho\sigma]} - C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]} \,. \qquad \text{(A.17)}
\end{aligned}
$$

Under $\rho \leftrightarrow \mu$ and $\sigma \leftrightarrow \nu$ in the above, I get

$$
\begin{aligned}
-(R_{\rho\sigma\mu\nu} - R_{\mu\nu\rho\sigma}) =\; & R_{\nu\sigma\mu\rho} - R_{\mu\rho\nu\sigma} - \nabla_{[\rho}C_{|\nu|\sigma\mu]} - C_{\nu\lambda[\rho}C^{\lambda}{}_{\sigma\mu]} \\
& - \nabla_{[\sigma}C_{|\rho|\mu\nu]} - C_{\rho\lambda[\sigma}C^{\lambda}{}_{\mu\nu]} \\
=\; & R_{\sigma\nu\rho\mu} - R_{\rho\mu\sigma\nu} - \nabla_{[\rho}C_{|\nu|\sigma\mu]} - C_{\nu\lambda[\rho}C^{\lambda}{}_{\sigma\mu]} \\
& - \nabla_{[\sigma}C_{|\rho|\mu\nu]} - C_{\rho\lambda[\sigma}C^{\lambda}{}_{\mu\nu]} \,. \qquad \text{(A.18)}
\end{aligned}
$$

Subtracting Eq. (A.18) from Eq. A.17 produces

$$R_{\rho\sigma\mu\nu} - R_{\mu\nu\rho\sigma} = \frac{1}{2}\Big(\nabla_{[\rho}C_{|\nu|\sigma\mu]} + C_{\nu\lambda[\rho}C^{\lambda}{}_{\sigma\mu]} + \nabla_{[\sigma}C_{|\rho|\mu\nu]} + C_{\rho\lambda[\sigma}C^{\lambda}{}_{\mu\nu]}$$
$$-\nabla_{[\mu}C_{|\sigma|\nu\rho]} - C_{\sigma\lambda[\mu}C^{\lambda}{}_{\nu\rho]} - \nabla_{[\nu}C_{|\mu|\rho\sigma]} - C_{\mu\lambda[\nu}C^{\lambda}{}_{\rho\sigma]}\Big)\,. \quad \text{(A.19)}$$

A.3.4 *Second Bianchi identity*

Replacing w_σ by $\nabla_\lambda w_\sigma$ in Eq. (A.12) and using Eq. (A.13), I get

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\nabla_\lambda w_\sigma = -R^{\rho}{}_{\lambda\mu\nu}\nabla_\rho w_\sigma - R^{\rho}{}_{\sigma\mu\nu}\nabla_\lambda w_\rho - C^{\rho}{}_{\mu\nu}\nabla_\rho\nabla_\lambda w_\sigma\,. \quad \text{(A.20)}$$

Again, taking ∇_λ on Eq. (A.12) yields

$$\nabla_\lambda(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)w_\sigma = -(\nabla_\lambda R^{\rho}{}_{\sigma\mu\nu})w_\rho - R^{\rho}{}_{\sigma\mu\nu}\nabla_\lambda w_\rho - (\nabla_\lambda C^{\rho}{}_{\mu\nu})\nabla_\rho w_\sigma$$
$$-C^{\rho}{}_{\mu\nu}\nabla_\lambda\nabla_\rho w_\sigma\,. \quad \text{(A.21)}$$

Now, antisymmetrising Eq. (A.20) and Eq. (A.21) in $\mu\nu\lambda$, we see that left hand sides of both the equations become equal. Thus I get

$$R^{\rho}{}_{[\lambda\mu\nu]}\nabla_\rho w_\sigma + C^{\rho}{}_{[\mu\nu}\nabla_{|\rho|}\nabla_{\lambda]}w_\sigma = (\nabla_{[\lambda}R^{\rho}{}_{|\sigma|\mu\nu]})w_\rho + (\nabla_{[\lambda}C^{\rho}{}_{\mu\nu]})\nabla_\rho w_\sigma$$
$$+C^{\rho}{}_{[\mu\nu}\nabla_{\lambda]}\nabla_\rho w_\sigma\,. \quad \text{(A.22)}$$

Using Eq. (A.12), I can write

$$C^{\rho}{}_{[\mu\nu}\nabla_{|\rho|}\nabla_{\lambda]}w_\sigma = C^{\rho}{}_{[\mu\nu}\nabla_{\lambda]}\nabla_\rho w_\sigma - R^{\alpha}{}_{\sigma\rho[\lambda}C^{\rho}{}_{\mu\nu]}w_\alpha - C^{\alpha}{}_{\rho[\lambda}C^{\rho}{}_{\mu\nu]}\nabla_\alpha w_\sigma\,. \quad \text{(A.23)}$$

Using the above equation and first Bianchi identity in Eq. (A.22), I get

$$\nabla_{[\lambda}R^{\rho}{}_{|\sigma|\mu\nu]} = -R^{\rho}{}_{\sigma\alpha[\lambda}C^{\alpha}{}_{\mu\nu]}\,. \quad \text{(A.24)}$$

B

A DIFFERENT WAY TO LOOK AT THE CONFORMAL SCALAR

I can write the torsionful Ricci scalar as the torsion-free Ricci scalar plus the torsion terms as

$$R = \widehat{R} - 2\widehat{\nabla}_\nu C^{\mu}{}_{\mu}{}^{\nu} - C^{\mu}{}_{\mu\sigma}C^{\nu}{}_{\nu}{}^{\sigma} + \frac{1}{4}C^{\mu\nu\sigma}C_{\mu\nu\sigma} + \frac{1}{2}C^{\mu\nu\sigma}C_{\nu\mu\sigma} \equiv \widehat{R} - f(C)\,. \tag{B.1}$$

In the scalar field Lagrangian in the action of Eq. (4.56), I will replace $\widehat{R}$ in the $\widehat{R}\phi^2$ term with $R + f(C)$; $f(C)$ being the explicit torsion terms above. The total action in vierbein-Einstein-Palatini formalism, with this modification, now reads as

$$\begin{aligned} S[e,A,\phi] = \int |e| d^4x F^{IJ}_{\mu\nu} e^{\mu}_{I} e^{\nu}_{J}\left(\frac{1}{2\kappa} - \frac{\phi^2}{12}\right) - \int |e| d^4x \left[\frac{1}{2}e^{\mu}_{I}e^{\nu I}\partial_\mu\phi\partial_\nu\phi + \frac{1}{6}\phi^2\widehat{\nabla}_\mu C^{\alpha}{}_{\alpha}{}^{\mu}\right. \\ \left. + \frac{1}{12}\left(C^{\mu}{}_{\mu\sigma}C^{\nu}{}_{\nu}{}^{\sigma} - \frac{1}{4}C^{\mu\nu\sigma}C_{\mu\nu\sigma} - \frac{1}{2}C^{\mu\nu\sigma}C_{\nu\mu\sigma}\right)\phi^2\right]\,. \end{aligned} \tag{B.2}$$

The equations obtained from this are

$$\begin{aligned} \delta e^{\nu}_{J}: \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\phi^2\right. \\ \left. + \frac{1}{6}\left[g_{\mu\nu}\widehat{\nabla}_\sigma\widehat{\nabla}^\sigma\phi^2 - \widehat{\nabla}_\mu\widehat{\nabla}_\nu\phi^2\right]\right) + \kappa\tilde{f}(C), \end{aligned} \tag{B.3a}$$

$$\delta A^{IJ}_{\nu}: \quad A^{IJ}_{\mu} = \omega^{IJ}_{\mu}[e], \tag{B.3b}$$

$$\begin{aligned} \delta\phi: \quad \widehat{\nabla}_\mu\widehat{\nabla}^\mu\phi - \frac{1}{6}R\phi - \phi\left(\frac{1}{3}\widehat{\nabla}_\mu C^{\alpha}{}_{\alpha}{}^{\mu} + \frac{1}{6}C^{\mu}{}_{\mu\sigma}C^{\nu}{}_{\nu}{}^{\sigma} - \frac{1}{24}C^{\mu\nu\sigma}C_{\mu\nu\sigma}\right. \\ \left. - \frac{1}{12}C^{\mu\nu\sigma}C_{\nu\mu\sigma}\right) = 0\,. \end{aligned} \tag{B.3c}$$

Here $\tilde{f}(C)$ in Eq. (B.3a) includes terms that contain torsion explicitly which go away upon using Eq. (B.3b) which is the torsion-free condition. R becomes $\widehat{R}$ upon using the torsion-free condition and thus the scalar Eq. (4.67c) becomes

$$\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6}\widehat{R}\phi = 0\,. \tag{B.4}$$

The scalar field equation thus gets back the torsion-free form given by Eq. (4.5). The modification in Eq. (B.1) leads to vanishing torsion on-shell. Thus I can identify the equations with those in the usual metric formalism.

Although I have considered a formalism which is torsion-free on-shell, Eq. (B.3c) is rather a more general equation; it will remain invariant even if torsion is non-zero because in such a case the torsion terms will cancel those in the $R\phi$ term and the equation will essentially be reduced to Eq. (B.4).

C

Λ AS AUXILIARY FIELD

I can decompose the total spin connection A_μ^{IJ} in terms the torsion-free component ω_μ^{IJ} and contorsion component Λ_μ^{IJ} as

$$A_\mu^{IJ} = \omega_\mu^{IJ} + \Lambda_\mu^{IJ} \,, \tag{C.1}$$

I can write the curvature as

$$F_{\mu\nu}^{IJ}(A) = \widehat{F}_{\mu\nu}^{IJ}(\omega) + \partial_\mu \Lambda_\nu^{IJ} - \partial_\nu \Lambda_\mu^{IJ} + \omega_{\mu K}^{I} \Lambda_\nu^{KJ} - \omega_{\nu K}^{I} \Lambda_\mu^{KJ} + \Lambda_{\mu K}^{I} \omega_\nu^{KJ} - \Lambda_{\nu K}^{I} \omega_\mu^{KJ} \\ + \Lambda_{\mu K}^{I} \Lambda_\nu^{KJ} - \Lambda_{\nu K}^{I} \Lambda_\mu^{KJ} \,. \tag{C.2}$$

Let us see the contribution of the terms linear in Λ in the vierbein-Einstein-Palatini action. I consider the integral

$$I = \int |e| d^4x (\partial_\mu \Lambda_\nu^{IJ} - \partial_\nu \Lambda_\mu^{IJ} + \omega_{\mu K}^{I} \Lambda_\nu^{KJ} - \omega_{\nu K}^{I} \Lambda_\mu^{KJ} + \Lambda_{\mu K}^{I} \omega_\nu^{KJ} - \Lambda_{\nu K}^{I} \omega_\mu^{KJ}) e_I^\mu e_J^\nu \,. \tag{C.3}$$

Considering the first, third and the sixth terms from above and denoting them as $\mathbb{A}$ together,

$$\begin{aligned} \mathbb{A} = &\ e_I^\mu e_J^\nu \partial_\mu \Lambda_\nu^{IJ} + e_I^\mu e_J^\nu \omega_{\mu K}^{I} \Lambda_\nu^{KJ} - e_I^\mu e_J^\nu \Lambda_{\nu K}^{I} \omega_\mu^{KJ} \\ = &\ \partial_\mu (e_I^\mu e_J^\nu \Lambda_\nu^{IJ}) - \Lambda_\nu^{IJ} e_I^\mu \partial_\mu e_J^\nu - \Lambda_\nu^{IJ} e_J^\nu \partial_\mu e_I^\mu + e_I^\mu e_J^\nu \omega_{\mu K}^{I} \Lambda_\nu^{KJ} - e_I^\mu e_J^\nu \Lambda_{\nu K}^{I} \omega_\mu^{KJ} \,. \end{aligned} \tag{C.4}$$

I will call $e_I^\mu e_J^\nu \Lambda_\nu^{IJ} \equiv \Lambda^\mu$. Thus

$$\mathbb{A} = \partial_\mu \Lambda^\mu - \Lambda_\nu^{IJ} e_I^\mu \partial_\mu e_J^\nu - \Lambda_\nu^{IJ} e_J^\nu \partial_\mu e_I^\mu + e_I^\mu e_J^\nu \omega_{\mu K}^{I} \Lambda_\nu^{KJ} - e_I^\mu e_J^\nu \Lambda_{\nu K}^{I} \omega_\mu^{KJ} \,. \tag{C.5}$$

Considering the first, third and fourth term from Eq. (C.5), I have

$$\begin{aligned}
\mathbb{A}.1 = &\ \partial_\mu \Lambda^\mu - \Lambda_\nu^{IJ} e_J^\nu \partial_\mu e_I^\mu + e_I^\mu e_J^\nu \omega^I_{\mu K} \Lambda_\nu^{KJ} \\
= &\ \partial_\mu \Lambda^\mu - \Lambda_\nu^{KJ} e_J^\nu e_K^\alpha e_\alpha^I \partial_\mu e_I^\mu + e_I^\mu e_J^\nu \omega^I_{\mu K} e_\alpha^K e_L^\alpha \Lambda_\nu^{LJ} \\
= &\ \partial_\mu \Lambda^\mu + \Lambda^\alpha (e_I^\mu \partial_\mu e_\alpha^I + \omega^I_{\mu K} e_\alpha^K e_I^\mu) \\
= &\ \partial_\mu \Lambda^\mu + \Lambda^\alpha \widehat{\Gamma}^\mu_{\mu\alpha} \\
= &\ \widehat{\nabla}_\mu \Lambda^\mu \,.
\end{aligned} \tag{C.6}$$

The remaining terms of $\mathbb{A}$ are

$$\begin{aligned}
\mathbb{A}.2 = &- \Lambda_\nu^{IJ} e_I^\mu \partial_\mu e_J^\nu - e_I^\mu e_J^\nu \Lambda^I_{\nu K} \omega_\mu^{KJ} \\
= &- \Lambda_\nu^{IK} e_I^\mu e_K^\alpha e_\alpha^J \partial_\mu e_J^\nu - e_I^\mu e_J^\nu \Lambda^I_{\nu L} e_{\alpha K} e^{\alpha L} \omega_\mu^{KJ} \\
= &\ \Lambda_\nu^{IK} e_I^\mu e_K^\alpha e_J^\nu \partial_\mu e_\alpha^J + \Lambda_\nu^{IL} e_I^\mu e_L^\alpha \omega^J_{\mu K} e_\alpha^K e_J^\nu \\
= &\ \Lambda_\nu^{IK} e_I^\mu e_K^\alpha (e_J^\nu \partial_\mu e_\alpha^J + \omega^J_{\mu K} e_\alpha^K e_J^\nu) \\
= &\ \Lambda_\nu^{IK} e_I^\mu e_K^\alpha \widehat{\Gamma}^\nu_{\mu\alpha} \\
= &\ \frac{1}{2} \Lambda_\nu^{IK} (e_I^\mu e_K^\alpha - e_I^\alpha e_K^\mu) \widehat{\Gamma}^\nu_{\mu\alpha} \\
= &\ 0 \,.
\end{aligned} \tag{C.7}$$

Here I have used the antisymmetry of Λ and the symmetry of $\widehat{\Gamma}$. Denoting the second, fourth and fifth terms of the integral I as $\mathbb{B}$ and proceeding the same way as above, I find that

$$\mathbb{B} = \widehat{\nabla}_\mu \Lambda^\mu \,. \tag{C.8}$$

Taking the above calculations into consideration, I get

$$I = \int |e| d^4x \; 2\, \widehat{\nabla}_\mu \Lambda^\mu \,. \tag{C.9}$$

Because this is a total divergence, the above integral can be taken to the boundary and neglected compared to the other terms of the total vierbein-Einstein-Palatini action. The effective action is thus

$$S_{VEP} = \frac{1}{2\kappa} \int |e| d^4x \left(\widehat{F}^{IJ}_{\mu\nu} + \Lambda^I_{\mu K} \Lambda_\nu^{KJ} - \Lambda^I_{\nu K} \Lambda_\mu^{KJ} \right) e_I^\mu e_J^\nu \,. \tag{C.10}$$

This appears in Eq. (5.17).

D

γ-MATRICES

I am working in the $(-+++)$ signature and the properties of γ-matrices are different from those in the $(+---)$ signature. I will mention the effect of signature change at the end of this Appendix. The basic anti-commutator is

$$\{\gamma_I, \gamma_J\} = 2\eta_{IJ} \,. \tag{D.1}$$

Thus in the $(-+++)$ signature

$$\gamma_0^2 = -\mathbb{I} \,, \tag{D.2}$$
$$\gamma_i^2 = \mathbb{I}, \qquad (i = 1, 2, 3) \,. \tag{D.3}$$

The Hermiticiy of the γ-matrices is as follows

$$\gamma_0^\dagger = -\gamma_0 \,, \tag{D.4}$$
$$\gamma_i^\dagger = \gamma_i \,, \tag{D.5}$$

which can be written in a compact form as

$$\gamma_I^\dagger = \gamma_0 \gamma_I \gamma_0 \,. \tag{D.6}$$

D.1 σ-MATRIX

The σ-matrix is defined as

$$\sigma_{IJ} = \frac{i}{2}[\gamma_I, \gamma_J] \,. \tag{D.7}$$

The Hermitian conjugate of the σ-matrix can be calculated as

$$\begin{aligned}
\sigma_{IJ}^{\dagger} &= -\frac{i}{2}[\gamma_I, \gamma_J]^{\dagger} \\
&= -\frac{i}{2}\left(\gamma_J^{\dagger}\gamma_I^{\dagger} - \gamma_I^{\dagger}\gamma_J^{\dagger}\right) \\
&= -\frac{i}{2}\left(\gamma_0\gamma_J\gamma_0^2\gamma_I\gamma_0 - \gamma_0\gamma_I\gamma_0^2\gamma_J\gamma_0\right) \\
&= -\frac{i}{2}\left(-\gamma_0\gamma_J\gamma_I\gamma_0 + \gamma_0\gamma_I\gamma_J\gamma_0\right) \\
&= -\frac{i}{2}\gamma_0[\gamma_I, \gamma_J]\gamma_0 \\
&= -\gamma_0\sigma_{IJ}\gamma_0 \,.
\end{aligned} \tag{D.8}$$

One useful relation is the commutator of γ and σ. In order to find the expression of the commutator note that

$$\begin{aligned}
\sigma_{IJ}\gamma_K &= \frac{i}{2}[\gamma_I, \gamma_J]\gamma_K \\
&= \frac{i}{2}\left(\gamma_I\gamma_J\gamma_K - \gamma_J\gamma_I\gamma_K\right) \\
&= \frac{i}{2}\left(2\eta_{JK}\gamma_I - \gamma_I\gamma_K\gamma_J - 2\eta_{IK}\gamma_J + \gamma_J\gamma_K\gamma_I\right) \\
&= \frac{i}{2}\left(4\eta_{JK}\gamma_I - 4\eta_{IK}\gamma_J + \gamma_K\gamma_I\gamma_J - \gamma_K\gamma_J\gamma_I\right) \\
&= 2i\left(\eta_{JK}\gamma_I - \eta_{IK}\gamma_J\right) + \gamma_K\sigma_{IJ} \,.
\end{aligned} \tag{D.9}$$

Thus I get

$$[\gamma_K, \sigma_{IJ}] = 2i(\eta_{IK}\gamma_J - \eta_{JK}\gamma_L) \,. \tag{D.10}$$

D.2 FIFTH γ-MATRIX

One more useful quantity is the fifth gamma matrix γ_5 defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \frac{i}{4!}\epsilon^{IJKL}\gamma_I\gamma_J\gamma_K\gamma_L \,. \tag{D.11}$$

It is easy to see from the above definition that γ_5 has the properties

$$\gamma_5^{\dagger} = \gamma_5 \,, \tag{D.12}$$

$$\gamma_5^2 = \mathbb{I} \,. \tag{D.13}$$

Some identities involving γ_5 are

$$\{\gamma_I, \gamma_5\} = 0\,, \tag{D.14}$$

$$\{\gamma_K, \sigma_{IJ}\} = 2\epsilon_{IJKL}\gamma^L\gamma_5\,, \tag{D.15}$$

$$[\sigma_{IJ}, \gamma_5] = 0\,. \tag{D.16}$$

D.3 CHANGE OF SIGNATURE

Under the change of signature from $(-+++)$ to $(+---)$ which is the more common signature used in Quantum Field Theory textbooks, I need to multiply the γ-matrices with $-i$, i.e., $\gamma_I \to -i\gamma_I$. In the signature $(+---)$ properties of the γ-matrices are given by

$$\gamma_0^2 = \mathbb{I}\,, \tag{D.17}$$

$$\gamma_i^2 = -\mathbb{I}\,, \tag{D.18}$$

$$\gamma_0^\dagger = \gamma_0\,, \tag{D.19}$$

$$\gamma_i^\dagger = -\gamma_i\,. \tag{D.20}$$

The Hermiticity of the γ-matrices is still given in the compact form of Eq. (D.6). γ_5 and its properties are invariant under the signature change. The σ-matrix is defined in the same as in Eq. (D.7). Let us see how this affects the Hermitian conjugate of the σ-matrix.

$$\begin{aligned}
\sigma_{IJ}^\dagger &= -\frac{i}{2}[\gamma_I, \gamma_J]^\dagger \\
&= -\frac{i}{2}\left(\gamma_J^\dagger\gamma_I^\dagger - \gamma_I^\dagger\gamma_J^\dagger\right) \\
&= -\frac{i}{2}\left(\gamma_0\gamma_J\gamma_0^2\gamma_I\gamma_0 - \gamma_0\gamma_I\gamma_0^2\gamma_J\gamma_0\right) \\
&= -\frac{i}{2}\left(\gamma_0\gamma_J\gamma_I\gamma_0 - \gamma_0\gamma_I\gamma_J\gamma_0\right) \\
&= \frac{i}{2}\gamma_0[\gamma_I, \gamma_J]\gamma_0 \\
&= \gamma_0\sigma_{IJ}\gamma_0\,.
\end{aligned} \tag{D.21}$$

The commutator $[\gamma_K, \sigma_{IJ}]$ however, remains unaffected under the change of signature. This because the calculations shown in Eq. (D.9) are independent of the signature of the metric.

E

CHIRAL SYMMETRY BREAKING AND CURRENT CONSERVATION

Let us see how geometrical breaking of chiral symmetry affects the current conservation of electron and neutrino. I will consider the left handed electron-neutrino doublet $\Psi_{eL} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ with torsion coupling constant λ_{eL} and right handed electron singlet e_R with coupling constant λ_{eR}. Different coupling constants for right handed and left handed components imply that chiral symmetry is broken here. The spinor covariant derivatives are given as

$$D_\mu \psi_{eL} = \partial_\mu \psi_{eL} - \frac{i}{4}\omega_\mu^{IJ}\sigma_{IJ}\psi_{eL} - \frac{i}{4}\lambda_{eL}\Lambda_\mu^{IJ}\sigma_{IJ}\psi_{eL}\,, \tag{E.1}$$

$$D_\mu e_R = \partial_\mu e_R - \frac{i}{4}\omega_\mu^{IJ}\sigma_{IJ}e_R - \frac{i}{4}\lambda_{eR}\Lambda_\mu^{IJ}\sigma_{IJ}e_R\,. \tag{E.2}$$

The action of Eq. (5.18) is thus given by

$$\begin{aligned} S = \int |e| d^4x \quad \Big[& \tfrac{1}{2\kappa}\widehat{F}^{IJ}_{\mu\nu}(\omega)e^\mu_I e^\nu_J + \tfrac{i}{2}\Big(\bar{\psi}_{eL}\gamma^K e^\mu_K\, \widehat{D}^f_\mu \psi_{eL} - (\bar{\psi}_{eL}\gamma^K e^\mu_K\, \widehat{D}^f_\mu \psi_{eL})^\dagger \\ & + \bar{e}_R\gamma^K e^\mu_K\, \widehat{D}^f_\mu e_R - (\bar{e}_R\gamma^K e^\mu_K\, \widehat{D}^f_\mu e_R)^\dagger\Big) + \tfrac{1}{2\kappa}\eta_{KL}\Lambda^{IK}_{[\mu}\Lambda^{LJ}_{\nu]}e^\mu_I e^\nu_J \\ & + \tfrac{1}{8}\Lambda^{IJ}_\mu e^\mu_K \left(\lambda_{eL}\bar{\psi}_{eL}\{\gamma^K,\sigma_{IJ}\}\psi_{eL} + \lambda_{eR}\bar{e}_R\{\gamma^K,\sigma_{IJ}\}e_R\right)\Big]\,. \end{aligned} \tag{E.3}$$

I have not considered mass terms here as they do not affect the calculations here. The on-shell expression for Λ is obtained as

$$\Lambda^{IJ}_\mu = \frac{\kappa}{8}e^K_\mu \left(\bar{\psi}_{eL}\{\gamma_K,\sigma^{IJ}\}\psi_{eL} + e_R\{\gamma_K,\sigma^{IJ}\}e_R\right)\,. \tag{E.4}$$

The Dirac equations of the doublet and singlet with the above expression of Λ are obtained respectively as

$$\begin{aligned}\gamma^K e^\mu_K \partial_\mu \psi_{eL} - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\gamma_K \sigma_{IJ}\psi_{eL} - \frac{3i\kappa}{8}\lambda^2_{eL}\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL}\gamma^I\gamma_5\psi_{eL} \\ - \frac{3i\kappa}{8}\lambda_{eL}\lambda_{eR}\bar{e}_R\gamma_I\gamma_5 e_R\gamma^I\gamma_5\psi_{eL} = 0\,,\end{aligned} \tag{E.5}$$

$$\begin{aligned}\gamma^K e^\mu_K \partial_\mu e_R - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\gamma_K \sigma_{IJ} e_R - \frac{3i\kappa}{8}\lambda^2_{eR}\bar{e}_R\gamma_I\gamma_5 e_R\gamma^I\gamma_5 e_R \\ - \frac{3i\kappa}{8}\lambda_{eL}\lambda_{eR}\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL}\gamma^I\gamma_5 e_R = 0\,.\end{aligned} \tag{E.6}$$

I will first find the equation of motion for electron $e = e_L + e_R$. For this purpose I add electron component of Eq. (E.5) and Eq. (E.6).

$$\begin{aligned}\gamma^K e^\mu_K \partial_\mu e - \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\gamma_K \sigma_{IJ} e - \frac{3i\kappa}{8}\lambda_{eL}\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL}\gamma^I\gamma_5(\lambda_{eL}e_L + \lambda_{eR}e_R) \\ - \frac{3i\kappa}{8}\lambda_{eR}\bar{e}_R\gamma_I\gamma_5 e_R\gamma^I\gamma_5(\lambda_{eL}e_L + \lambda_{eR}e_R) = 0\,.\end{aligned} \tag{E.7}$$

The conjugate equation is obtained as

$$\begin{aligned}\partial_\mu \bar{e}\gamma^K e^\mu_K + \frac{i}{4}\omega^{IJ}_\mu e^{\mu K}\bar{e}\sigma_{IJ}\gamma_K + \frac{3i\kappa}{8}\lambda_{eL}\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL}(\lambda_{eL}\bar{e}_L + \lambda_{eR}\bar{e}_R)\gamma^I\gamma_5 \\ + \frac{3i\kappa}{8}\lambda_{eR}\bar{e}_R\gamma_I\gamma_5 e_R(\lambda_{eL}\bar{e}_L + \lambda_{eR}\bar{e}_R)\gamma^I\gamma_5 = 0\,.\end{aligned} \tag{E.8}$$

Here I have used

$$\begin{aligned}(\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL})^\dagger &= \psi^\dagger_{eL}\gamma^\dagger_5\gamma^\dagger_I\gamma^\dagger_0\psi_{eL} \\ &= -\psi^\dagger_{eL}\gamma_5\gamma_0\gamma_I\gamma_0\gamma_0\psi_{eL} \\ &= -\bar{\psi}_{eL}\gamma_5\gamma_I\psi_{eL} \\ &= \bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL}\,.\end{aligned} \tag{E.9}$$

Pre-multiplying Eq. (E.7) by $\bar{e}$ and post-multiplying Eq. (E.8) by e and proceeding in the same way as in Eq. (3.38), I get

$$\begin{aligned}\widehat{\nabla}_\mu J^\mu_e + \frac{3i\kappa}{4}\left(\lambda_{eR}\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL} + \lambda_{eR}\bar{e}_R\gamma_I\gamma_5 e_R\right)\Big[(\lambda_{eL}\bar{e}_L + \lambda_{eR}\bar{e}_R)\gamma^I\gamma_5(e_L + e_R) \\ -(\bar{e}_L + \bar{e}_R)\gamma^I\gamma_5(\lambda_{eL}e_L + \lambda_{eR}e_R)\Big] = 0\,,\end{aligned} \tag{E.10}$$

where $J_e^\mu = \bar{e}\gamma^I e e_I^\mu$ is the electron current. Now,

$$(\lambda_{eL}\bar{e}_L + \lambda_{eR}\bar{e}_R)\gamma^I\gamma_5(e_L + e_R) - (\bar{e}_L + \bar{e}_R)\gamma^I\gamma_5(\lambda_{eL}e_L + \lambda_{eR}e_R)$$
$$= (\lambda_{eL} - \lambda_{eR})\left(\bar{e}_L\gamma^I\gamma_5 e_R - \bar{e}_R\gamma^I\gamma_5 e_L\right). \tag{E.11}$$

Using

$$e_L = \frac{1}{2}(1-\gamma_5)e\,, \tag{E.12}$$
$$e_R = \frac{1}{2}(1+\gamma_5)e\,, \tag{E.13}$$

I get

$$\begin{aligned}\bar{e}_L\gamma^I\gamma_5 e_R &= \frac{1}{4}\overline{(1-\gamma_5)e}\gamma^I\gamma_5(1+\gamma_5)e \\ &= \frac{1}{4}((1-\gamma_5)e)^\dagger\gamma_0\gamma^I\gamma_5(1+\gamma_5)e \\ &= \frac{1}{4}e^\dagger(1-\gamma_5)\gamma_0\gamma^I\gamma_5(1+\gamma_5)e \\ &= \frac{1}{4}\bar{e}\gamma^I\gamma_5(1+\gamma_5)e + \frac{1}{4}\bar{e}\gamma_5\gamma^I\gamma_5(1+\gamma_5)e \\ &= \frac{1}{4}\bar{e}\gamma^I\gamma_5(1+\gamma_5)e - \frac{1}{4}\bar{e}\gamma^I\gamma_5^2(1+\gamma_5)e \\ &= 0\,. \end{aligned} \tag{E.14}$$

Here I have used the properties of gamma matrices. Similarly I can show that

$$\bar{e}_R\gamma^I\gamma_5 e_L = 0\,. \tag{E.15}$$

Thus the contribution of torsion terms still vanishes in spite of the coupling constants and equation is simply given by

$$\widehat{\nabla}_\mu J_e^\mu = 0\,. \tag{E.16}$$

I will also investigate the current conservation of neutrino. For this purpose I consider the neutrino component of Eq. (E.5) i.e.,

$$\gamma^K e_K^\mu\partial_\mu\nu_L - \frac{i}{4}\omega_\mu^{IJ}e^{\mu K}\gamma_K\sigma_{IJ}\nu_L - \frac{3i\kappa}{8}\lambda_{eL}^2\bar{\psi}_{eL}\gamma_I\gamma_5\psi_{eL}\gamma^I\gamma_5\nu_L$$
$$-\frac{3i\kappa}{8}\lambda_{eL}\lambda_{eR}\bar{e}_R\gamma_I\gamma_5 e_R\gamma^I\gamma_5\nu_L = 0\,. \tag{E.17}$$

The conjugate equation is obtained as

$$\partial_\mu \bar{\nu}_L \gamma^K e^\mu_K + \frac{i}{4}\omega^{IJ}_\mu e^{\mu K} \bar{\nu}_L \sigma_{IJ} \gamma_K + \frac{3i\kappa}{8}\lambda^2_{eL} \bar{\psi}_{eL} \gamma_I \gamma_5 \psi_{eL} \bar{\nu}_L \gamma^I \gamma_5 \\ + \frac{3i\kappa}{8}\lambda_{eL}\lambda_{eR} \bar{e}_R \gamma_I \gamma_5 e_R \bar{\nu}_L \gamma^I \gamma_5 = 0\,. \tag{E.18}$$

Pre-multiplying Eq. (E.17) with $\bar{\nu}_L$ and post-multiplying Eq. (E.18) with ν_L and adding them together we can see that the non-linear terms get cancelled. The resulting equation is thus

$$\widehat{\nabla}_\mu J^\mu_{\nu_L} = 0\,, \tag{E.19}$$

where $J^\mu_{\nu_L} = \bar{\nu}_L \gamma^I \nu_L e^\mu_I$ is the neutrino current. I can thus conclude that geometrical breaking of chiral symmetry does not affect the current conservation of electron $e = e_L + e_R$, nor of the neutrinos.

F

FIERZ IDENTITIES

From a given spinor ψ, we have 16 different bilinears [20] namely

$$\text{Scalar: } S = \bar{\psi}\psi\,, \tag{F.1}$$

$$\text{Pseudoscalar: } P = i\bar{\psi}\gamma_5\psi\,, \tag{F.2}$$

$$\text{Vector: } V_I = i\bar{\psi}\gamma_I\psi\,, \tag{F.3}$$

$$\text{Axial vector: } A_I = i\psi\gamma_5\gamma_I\psi\,, \tag{F.4}$$

$$\text{Tensor: } T_{IJ} = \bar{\psi}\sigma_{IJ}\psi\,. \tag{F.5}$$

The independent identities of these bilinears are

$$1. T_{IJ}V^J = -PA_I\,, \tag{F.6}$$

$$2. {}^*T_{IJ}V^J = SA_J;\,({}^*T_{IJ} = -\frac{i}{2}\epsilon_{IJKL}T^{KL})\,, \tag{F.7}$$

$$3. V_IV^I = -A_IA^I = -(S^2+P^2)\,. \tag{F.8}$$

I have mainly used the last identity in the following form in my thesis.

$$(\bar{\psi}\gamma_I\gamma_5\psi)(\bar{\psi}\gamma^I\gamma_5\psi) = (\bar{\psi}\gamma_5\psi)^2 - (\bar{\psi}\psi)^2\,. \tag{F.9}$$

Usually Fierz identities written for more than one species of spinors and they are connected to the reordering of the spinor fields in a four fermion interaction. Suppose I have four spinors $\psi_1,\ \psi_2,\ \psi_3,\ \psi_4$ and I consider an interaction term given by

$$(\bar{\psi}_1A\psi_2)(\bar{\psi}_3B\psi_4)\,. \tag{F.10}$$

This interaction can also be written as

$$(\bar{\psi}_1 M \psi_4)(\bar{\psi}_3 N \psi_2)\,. \tag{F.11}$$

Fierz identities relate the matrices $A,\ B$ to $M,\ N$. The matrices are usually written in terms of the basis

$$\{\Gamma^A\} = \{\mathbb{I}, \gamma_5, \gamma^I, \gamma_5\gamma^I, \sigma^{IJ}\} \qquad (I, J = 0, 1, 2, 3)\,. \tag{F.12}$$

There are thus 16 different bilinears which are classified into different classes according to their behaviour under Lorentz transformation. For discussion and derivation of the identities see [22, 109].

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